# Homology and K-theory of torsion free ample groupoids

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# ample groupoids

a Hausdorff topological groupoid $G$ is *ample* if:

1. base space $X = G^{(0)}$ is totally disconnected
2. range and sournce maps $r, s\colon G \to X$ are local homeomorphisms

**Example (transformation groupoid)**

$\Gamma \curvearrowright X$: Cantor dynamical system $\rightsquigarrow G = \Gamma \ltimes X$; $s(\gamma, x) = x, r(\gamma, x) = \gamma x$

invariants

- groupoid C*-algebras $C^*_r G$ (reduced), $C^* G$ (full)
- their K-theoretic invariants $K_i(C^*_r G)$, $K^i(C^*_r G)$, …
- groupoid homology $H_i(G, \mathbb{Z})$, cohomology $H^i(G, \mathbb{Z})$, …

Matui: $H_i(G, \mathbb{Z})$ and $K_i(C^*_r G)$ are similar; maybe $H_{i+2\mathbb{Z}}(G, \mathbb{Z}) \simeq K_i(C^*_r G)$?

# groupoid homology

to define *homology* $H_\bullet(G, \mathbb{Z})$ of ample groupoid $G$:

1. take nerves
$$G^{(k)} = \{(g_1, \dots, g_k) \mid sg_j = rg_{j+1}\}$$
2. simplicial étale space structure $(G^{(n)})_{n=0}^\infty$
   - concatenate: $d_i \colon G^{(n)} \to G^{(n-1)}$, $(\dots, g_{i-1}g_i, \dots)$ (or drop $g_1$ or $g_n$)
   - insert unit: $s_i \colon G^{(n)} \to G^{(n+1)}$, $(\dots, g_i, \mathrm{id}_{sg_i}, g_{i+1}, \dots)$
3. summation along fibers $d_{i*} \colon C_c(G^{(n)}, \mathbb{Z}) \to C_c(G^{(n-1)}, \mathbb{Z})$, etc.
4. $H_\bullet(G, \mathbb{Z})$: homology of $d = \sum_i (-1)^i d_{i*}$

generalization: Crainic–Moerdijk theory

- $G$: arbitrary étale groupoid
- coefficient: complex of $G$-sheaves $M_\bullet$ (in place of $\underline{\mathbb{Z}}$)
- $M'_\bullet$: c-soft resolution of $M_\bullet$

*hyperhomology* $\mathbb{H}_k(G, M_\bullet)$ as homology of bicomplex $\Gamma_c(G^{(n)}, M'_k)$

# groupoid homology to K-theory

**Theorem (P.–Y.)**

*$G$: ample groupoid such that*

- *stabilizers $G_x^x = \{g \mid sg = rg = x\}$ are torsion free*
- *satisfies the strong Baum–Connes conjecture (e.g., amenable)*

*$A$: separable $G$-$C^*$-algebra*

*$\Rightarrow \exists$ spectral sequence*

$$E^2_{p,q} = H_p(G, K_q(A)) \Rightarrow K_{p+q}(C^*G \ltimes_r A)$$

prototype: Pimsner–Voiculescu sequence for $\alpha\colon \mathbb{Z} \curvearrowright A$

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{1-\alpha_*} & K_0(A) & \longrightarrow & K_0(\mathbb{Z} \ltimes_\alpha A) \\ \uparrow & & & & \downarrow \\ K_1(\mathbb{Z} \ltimes_\alpha A) & \longleftarrow & K_1(A) & \xleftarrow[1-\alpha_*]{} & K_1(A) \end{array}$$

# groupoid homology to K-theory

**Theorem (P.–Y.)**

$\exists$ *spectral sequence*

$$E^2_{p,q} = H_p(G, K_q(A)) \Rightarrow K_{p+q}(C^*G \ltimes_r A)$$

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$\rightsquigarrow$ extension

$$H_0(\mathbb{Z}, K_*(A)) = K_*(A)_{\alpha\text{-coinv}} \to K_*(\mathbb{Z} \ltimes_\alpha A) \to K_{*+1}(A)^{\alpha\text{-inv}} = H_1(\mathbb{Z}, K_{*+1}(A))$$

# Smale space

Smale space $(Y, \psi)$:

- $Y$: compact metric space
- $\psi$: "hyperbolic" dynamics on $Y$ with scale $0 < \lambda < 1$
- *local stable set* (contracting direction):
  $Y^s(y, \varepsilon) = \{y' \mid d(y, y') < \varepsilon, d(\psi^n y, \psi^n y') \leq \lambda^n d(y, y'), n \in \mathbb{N}\}$
- *unstable equivalence* relation $R^u(Y, \psi)$:
  $y \sim_u y' \Leftrightarrow d(\psi^{-n} y, \psi^{-n} y') \leq \lambda^{-n} d(y, y')$

## Example

- $Y$ totally disconnected ≡ shift of finite type (symbolic dynamics)
- solenoid $\varprojlim \mathbb{T} = \{(z_0 = z_1^m, z_1 = z_2^m, \dots)\}$, shift map
- substitution tiling system (Anderson–Putnam)

# Putnam homology

$(Y,\psi)$: Smale space with totally disconnected stable sets
Putnam (cf. Bowen): open Markov partitions give

- $(\Sigma,\sigma)$: shift of finite type
- $f\colon (\Sigma,\sigma) \to (Y,\psi)$ homeomorphism on stable sets (s-bijective)

Putnam homology $H^s_\bullet(Y,\psi) = H_\bullet(D^s(\Sigma_\bullet))$ from

- simplicial shit of finite type

$$\Sigma_L = \underbrace{\Sigma \times_Y \cdots \times_Y \Sigma}_{(L+1)}, \qquad \sigma_L = (\sigma \times \cdots \times \sigma)|_{\Sigma_L}$$

- simplicial group $D^s(\Sigma_L)$: Krieger's dimension group

P.–Y.: spectral sequence

1. $H^s_p(Y,\psi) \otimes K_q(\mathbb{C}) \Rightarrow K_{p+q}(C^* R^u(Y,\psi))$
2. analogue for unstable Ruelle algebra $R_u = \mathbb{Z} \ltimes_\psi R^u(Y,\psi)$

# why such correspondence?

simplicial $G$-space $(G^{(n+1)})_{n=0}^{\infty}$ (not $(G^{(n)})_{n=0}^{\infty}$!)

- concatenate $i$-th and $(i+1)$-th (or drop last): $d_i \colon G^{(n+1)} \to G^{(n)}$
- insert unit at $(i+1)$-th: $s_i \colon G^{(n+1)} \to G^{(n+2)}$

$G$ étale $\Rightarrow$ simplicial object $(C_0(G^{(n+1)}))_{n=0}^{\infty}$ in $KK^G$

- $P_n = C_0(G^{(n+1)}) \otimes A$ forming a *resolution* of $A$ in $KK^G$
- $E^1_{pq} = F(C_0(\mathbb{R}^q) \otimes P_p) \Rightarrow F(A)$ for good functor $F$ on $KK^G$

$G$ ample $\Rightarrow$ defining complex of $H_\bullet(G, K_i(A))$ from

$$K_i(G \ltimes P_n) \simeq C_c(G^{(n)}, K_i(A))$$

same scheme works for $H < G$ with $H^{(0)} = X = G^{(0)}$

$$P_n = (\mathrm{Ind}_H^G \mathrm{Res}_H^G)^{n+1} A$$

with simplicial structure from $KK^G(\mathrm{Ind}_H^G A, B) \simeq KK^H(A, \mathrm{Res}_H^G B)$

# triangulated category and spectral sequence

Christensen, Meyer–Nest, Meyer

- $\mathscr{S}, \mathscr{T}$: triangulated categories, suspension $\Sigma$
- $F \colon \mathscr{T} \to \mathscr{S}$: exact functor
- $\mathscr{I}(A, B) = \ker F|_{\mathscr{T}(A,B)} \lhd \mathscr{T}(A, B)$: homological ideal
- $(P_n)_{n=0}^{\infty}$: $\mathscr{I}$-projective resolution of $A$
  1. $\mathscr{S}(P, N) = 0$ when $F(N) = 0$
  2. $\dots \to F(P_1) \to F(P_0) \to F(A)$ split exact

$\rightsquigarrow$ approximation $P_A \in \langle \mathscr{I}\text{-proj} \rangle$; ghost $N_A \in \mathscr{I}$

$$P_A \to A \to N_A \to \Sigma P_A \quad \text{exact triangle}$$

$K \colon \mathscr{T} \to \mathrm{Ab}$ homological functor $\rightsquigarrow$ spectral sequence

$$E^1_{p,q} = K(\Sigma^{-q} P_p) \Rightarrow K(\Sigma^{-(p+q)} P_A)$$

# triangulated category structure of $KK^G$

Le Gall: $KK^G(A, B)$ for

- Hausdorff locally compact groupoid $G$
- continuous action of $G$: $C_0(G) \otimes_{{}_s C_0(X)} A \to C_0(G) \otimes_{{}_r C_0(X)} A$
- $KK^G(A, B)$ for separable $G$-C*-algebras, composition
- descent $KK^G(A, B) \to KK(G \ltimes_r A, G \ltimes_r B)$

strict equivariantization of cycles (Oyono-Oyono) ⇒ 'Cuntz picture';

- $C^*_G \to KK^G$: universal functor with stability, homotopy invariance, split exactness
- $KK^G$ triangulated with mapping cone triangles

$$(A \to B \to C \to \Sigma A) \simeq (\Sigma B' \to Con(f) \to A' \to B')$$

goal: capture $KK^G \to \mathrm{Ab}, A \mapsto K_0(G \ltimes_r A)$

# projective resolution from adjoint pairs

- $\mathcal{S}, \mathcal{T}$: triangulated categories
- $E\colon \mathcal{S} \to \mathcal{T}$, $F\colon \mathcal{T} \to \mathcal{S}$: triangulated functors, with adjunction

$$\mathcal{S}(B, FA) \simeq \mathcal{T}(EB, A)$$

- $\mathcal{J} = \ker F$: homological ideal
- $A' = EB$ is $\mathcal{J}$-projective: $FA = 0 \Rightarrow \mathcal{T}(A', A) = 0$
- $L = EF\colon \mathcal{T} \to \mathcal{T}$ comonad (comonoid of endofunctor)

concrete setting: $\mathcal{T} = KK^G$, $\mathcal{S} = KK^X$, $F = \mathrm{Res}^G_X$, $E = \mathrm{Ind}^G_X$

**Proposition**

*simplicial object $(L^{n+1}A)_{n=0}^{\infty}$ forms an $\mathcal{J}$-projective resolution of $A \in \mathcal{T}$*

key idea: $FL^{\bullet+1}A \to FA \to 0$ exact in $\mathcal{S}$

# how to use Baum–Connes conjecture

**Theorem (Tu; groupoid version of Higson–Kasparov)**

*$G$: second countable Hausdorff locally compact groupoid with a proper cond. neg. definite function; (e.g., amenable) then $\exists$ proper $G$-space $Z$, $G \ltimes Z$-$C^*$-algebra $P$*

- *$P$ is a continuous field of nuclear $C^*$-algebras*
- *$P \simeq C_0(X)$ in $KK^G$ ($X = G^{(0)}$)*

**Proposition**

*$P_A \simeq A$ in $KK^G$ for*

- *$G$: étale, torsion free stabilizers, and as above*
- *$P_A$: approximation of $A$ from $\mathscr{I}$-projective objects for $\mathscr{I} = \ker(\mathrm{Res}^G_X \colon KK^G \to KK^X)$*

## summary

$G$: étale groupoid on $X$, $A$: $G$-C*-algebra

- comonad $L = \mathrm{Ind}_X^G \mathrm{Res}_X^G \colon KK^G \to KK^G$
- $(L^{n+1}A)_{n=0}^{\infty}$: simplicial approximation of $A$ by $(\ker \mathrm{Res}_X^G)$-projective objects
- $(K_*(G \ltimes_r L^{n+1}A))_{n=0}^{\infty}$: simplicial approximation of $K_*(G \ltimes_r A)$
- $K_*(G \ltimes_r L^{n+1}A) \simeq K_*(C_0(G^{(n)}) \otimes_{{}_s C_0(X)} A)$

if $G$ is ample (totally disconnected):

$$K_*(C_0(G^{(n)}) \otimes_{{}_s C_0(X)} A) \simeq C_c(G^{(n)}, \mathbb{Z}) \otimes_{C_c(X,\mathbb{Z})} K_*(A)$$

$\rightsquigarrow$ groupoid homology with coefficients $H_*(G, K_*(A))$:

# summary

$G$: ample groupoid on $X$, $A$: $G$-C*-algebra

- comonad $L = \operatorname{Ind}_X^G \operatorname{Res}_X^G \colon KK^G \to KK^G$
- $K_*(G \ltimes_r L^{n+1} A) \simeq K_*(C_0(G^{(n)}) \otimes_{s C_0(X)} A)$

$$K_*(C_0(G^{(n)}) \otimes_{s C_0(X)} A) \simeq C_c(G^{(n)}, \mathbb{Z}) \otimes_{C_c(X,\mathbb{Z})} K_*(A)$$

$\rightsquigarrow$ groupoid homology with coefficients $H_*(G, K_*(A))$:

**Theorem (P.–Y.)**

- *stabilizers $G_x^x = \{g \mid sg = rg = x\}$ are torsion free*
- *satisfies the consequence of Tu's theorem ($\exists$ proper $G$-space $Z$, $C_0(Z)$-nuclear $P$ such that $P \simeq_{KK^G} C_0(X)$)*

*$\exists$ spectral sequence*

$$E^2_{p,q} = H_p(G, K_q(A)) \Rightarrow K_{p+q}(G \ltimes_r A)$$

# induction from subgroupoid

Want: "induction" $\mathrm{Ind}_H^G \colon KK^H \to KK^G$ for

- $G$: étale groupoid with base $X = G^{(0)}$
- $H$: open subgroupoid with $X = H^{(0)}$

: $G/H$ might not be Hausdorff
working model: $\mathrm{Ind}_H^G A = (C_0(G) \otimes_{s C_0(X)} A) \rtimes H$
$(\mathrm{Ind}_H^G \mathrm{Res}_H^G)^n C_0(X) \rightsquigarrow$ groupoid pullback $H^{\times_G n}$:

arrows of $H^{\times_G n}$:

$$\begin{array}{ccccccc} x'_1 & \xleftarrow{g'_1} & x'_2 & \xleftarrow{g'_2} & \cdots & \xleftarrow{g'_{n-1}} & x'_n \\ {\scriptstyle h_1}\uparrow & & {\scriptstyle h_2}\uparrow & & & & {\scriptstyle h_n}\uparrow \\ x_1 & \xleftarrow[g_1]{} & x_2 & \xleftarrow[g_2]{} & \cdots & \xleftarrow[g_{n-1}]{} & x_n \end{array}$$

# transversality

want: understand $H^{\times_G n}$ for

- $(Y, \psi)$: Smale space with totally disconnected stable sets
- $X$: transversal for unstable equivalence relation (e.g., $Y^s(y, \varepsilon)$)
- $f\colon \Sigma \to Y$: factor map from SFT, bijective on stable sets
- $G = R^u(Y, \psi)|_X$, $H = R^u(\Sigma, \sigma)|_{f^{-1}X}$

**Proposition**

*given $a^1, \ldots, a^n \in \Sigma$, $f(a^k)$ mutually unstably equivalent then $\exists b^k$ unstably equivalent to $a^k$, $f(b^k)$ all equal*

**Corollary**

1. *Morita equivalence* $R^u(\Sigma, \sigma)^{\times_{R^u(Y,\psi)} n} \sim R^u(\Sigma^{\times_Y n}, \sigma^{\times n})$
2. $H_\bullet(K_0(G \ltimes (\mathrm{Ind}_H^G \mathrm{Res}_H^G)^{\bullet+1} C_0(X))) \simeq H^s_\bullet(Y, \psi)$

# homology for Ruelle algebra

Ruelle algebra $R_u(Y,\psi) = \mathbb{Z} \ltimes_\psi C^*R^u(Y,\psi)$
shift of finite type $(\Sigma_A,\sigma)$ for $A \in M_n(\mathbb{N}) \rightsquigarrow$ *Bowen–Franks groups*

- $\mathrm{BF}_0(\Sigma_A) = \mathrm{BF}(A) = \mathrm{cok}(1-A) \simeq K_0(R_u(\Sigma_A,\sigma))$
- $\mathrm{BF}_1(\Sigma_A) = \ker(1-A) \simeq K_1(R_u(\Sigma_A,\sigma))$

from $s$-bijective $(\Sigma,\sigma) \to (Y,\psi)$, $\Sigma_L = \Sigma \times_Y \cdots \times_Y \Sigma$ ($L+1$ times)

- étale groupoid model $\mathbb{Z} \ltimes G$ for $G = R^u(Y,\psi)|_X$; $\psi(X) = X$
- subgroupoid $\mathbb{Z} \ltimes H$ for $H = R^u(\Sigma,\sigma)|_{X'}$; $f(X') = X, \sigma(X') = X'$

**Theorem (P.–Y.)**

*$\exists$ spectral sequence $E^r_{pq} \Rightarrow K_{p+q}(C^*R_u(Y,\psi))$ with*

- $E^2_{pq} = 0$ *for odd* $q$
- $E^2_{pq} = E^2_{p0} \simeq \mathbb{H}_p(\mathbb{Z}, D^s(\Sigma_\bullet))$ *for even* $q$*, with long exact sequence*

$$\dots \to H_{p-1}(\mathrm{BF}_1(\Sigma_\bullet)) \to E^2_{p0} \to H_p(\mathrm{BF}_0(\Sigma_\bullet)) \to H_{p-2}(\mathrm{BF}_1(\Sigma_\bullet)) \to \dots$$