Hilbert transforms and Cotlar-type identities for groups acting on trees

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The Hilbert transform

Definition

For $f \in C_c^\infty(\mathbb{R})$,

$$(Hf)(x) = \mathrm{p.v.} \int_{\mathbb{R}} f(y) \frac{1}{x-y} dy.$$

The Hilbert transform as a Fourier multiplier:

$$\widehat{(Hf)}(\xi) = -i \,\mathrm{sgn}(\xi) \cdot \widehat{f}(\xi), \quad \xi \in \mathbb{R}.$$

Motivation: Convergence of Fourier series.

Problem

Let $f \in L_p(\mathbb{T})$ for $1 < p < \infty$. Do we have

$$\sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{2\pi ik\theta} \longrightarrow f(\theta) \text{ in } L_p\text{-norm?}$$

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Let $f \in L_p(\mathbb{T})$ for $1 < p < \infty$. Do we have

$$\lim_{N\to\infty} (T_{\mathbf{1}_{[-N,N]}} f)(\theta) = \lim_{N\to\infty} \sum_{k=-N}^{N} \widehat{f}(k) e^{2\pi i k\theta} \longrightarrow f(\theta) \text{ in } L_p\text{-norm?}$$

$$L_p\text{-norm convergence} \iff \sup_N \| T_{\mathbf{1}_{[-N,N]}} : L_p(\mathbb{T}) \to L_p(\mathbb{T}) \| < \infty.$$

Symbol	Multiplier
$i\,\mathrm{sgn}(k)$	H
$\mathbf{1}_{[0,\infty)}(k)$	$\frac{1}{2}(\mathbf{1}+iH)$
$\mathbf{1}_{[a,\infty)}(k)$	$\underbrace{\frac{1}{2}(\mathbf{1}+iM_{e^{-2\pi iax}}HM_{e^{2\pi iax}})}_{H_a}$
$\mathbf{1}_{[a,b]}(k)=\mathbf{1}_{[a,\infty)}(k)\cdot\mathbf{1}_{[-b,\infty)}(-k)$	$H_a\tilde{H}_b$

where $M_{f(x)}g(x)=f(x)g(x)$.

$$\|H:L_p(\mathbb{T})\to L_p(\mathbb{T})\|<\infty \iff \sup_N \|T_{\mathbf{1}_{[-N,N]}}:L_p(\mathbb{T})\to L_p(\mathbb{T})\|<\infty.$$

The boundedness of Hilbert transform on $\mathbb{R}$

Results:

- Unbounded on L_p for $p = 1, \infty$.
- Trivially bounded on L_2.
- **M. Riesz (1924)** Bounded for $1 < p < \infty$.
- **Cotlar (1955)** Recursive: $p = 2^k +$ Marcinkiewicz's Interpolation.
- **Kolmogorov (1924)** Weak L_1.
- **Calderón and A. Zygmund (1952)**.

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Classical Cotlar Identity:

$$(Hf)^2 = f^2 + 2H(f\,Hf).$$

Generalized by Mei and Ricard (2017) for amalgamated free product of von Neumann algebras.

Non-Abelian groups

G: discrete group.

Left regular representation

$\lambda : \mathrm{G} \to \mathcal{U}(\ell_2(\mathrm{G}))$ with $\lambda_g \varphi(h) = \varphi(g^{-1}h)$. $\mathcal{L}(\mathrm{G}) = \{\lambda_g\}''_{g\in\mathrm{G}}$.

Non-abelian Fourier transform

For $\widehat{f} \in \ell_1(\mathrm{G}), f := \sum_{\mathrm{G}} \widehat{f}(g)\lambda_g$ is a bounded linear map $\ell_2(\mathrm{G}) \to \ell_2(\mathrm{G})$.

Non-commutative L_p-spaces

$L_p(\widehat{\mathrm{G}}) := L_p(\mathcal{L}(\mathrm{G}), \tau) =$ "$\{f : \tau(|f|^p)^{\frac{1}{p}} < \infty\}$" with $\tau(f) = \widehat{f}(e)$.

Fourier multipliers on $\mathcal{L}(\mathrm{G})$:

$m \in \ell_\infty(\mathrm{G}) \rightsquigarrow T_m f := \sum_{\mathrm{G}} m(g)\widehat{f}(g)\lambda_g$.

Cotlar identity I

Problem

Does it hold that

$$\|H = T_m : L_p(\widehat{\mathrm{G}}) \to L_p(\widehat{\mathrm{G}})\| < \infty?$$

Theorem (Mei-Ricard '17, Cotlar '55)

Let $H : L_2(\widehat{\mathrm{G}}) \to L_2(\widehat{\mathrm{G}})$ be self-adjoint and bounded. If

$$H(x)H(x)^* = H(xH(x)^*) + H(xH(x)^*)^* - H(H(xx^*))^*, \ \text{(Cotlar identity)}$$

then $\|H : L_p(\widehat{\mathrm{G}}) \to L_p(\widehat{\mathrm{G}})\| \lesssim \left(\frac{p}{p-1}\right)^{\beta}$, $\beta = \log_2(1+\sqrt{2})$.

Cotlar identity II

Proposition (González Pérez-Parcet-X)

Let $H = T_m$. TFAE:

- Cotlar identity holds for H.
- Condition on the symbol:

$$(m(gh) - m(g))(m(g^{-1}) - m(h)) = 0 \qquad \text{(Cotlar condition)}$$

  for all $g, h \in G \backslash \{e\}$.

Proof.

$H(f)H(f)^* - H(fH(f)^*) - H(fH(f)^*)^* + H(H(ff^*))^* = 0$ for $f \in \mathbb{C}[G]$ is equivalent to

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$$\sum_{g,h \in G\setminus\{e\}} \left[m(gh)m(h) - m(g)m(h) - m(g^{-1})m(gh) + m(g)m(g^{-1}) \right] \widehat{f}(gh)\overline{\widehat{f}(h)}\lambda_g = 0.$$

□

Hilbert transforms for groups acting on trees

Let G be a group acting on a tree X *without inversion*.
Choose a vertex P_0 in X and write $X\backslash\{P_0\}$ as the disjoint union of its connected components $X\backslash\{P_0\} = \sqcup_i X_i$.

Define a bounded function on X by $\tilde{m}(P_0) = 0$ and $\tilde{m}(X_i) = C_i$, $C_i \neq C_j$ when $i \neq j$.
The function $\tilde{m}$ induces a function on G by $m(g) = \tilde{m}(g \cdot P_0)$ for any $g \in \mathrm{G}$.

Theorem (González Pérez-Parcet-X)

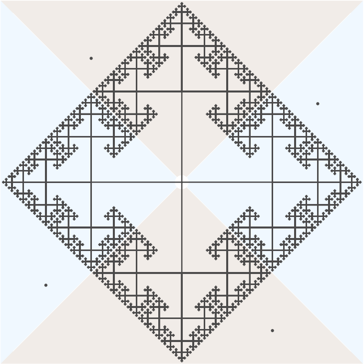
The function m defined above satisfies the Cotlar condition that for any $g, h \in \mathrm{G}$ s.t. $g \cdot P_0 \neq P_0$ and $h \cdot P_0 \neq P_0$,

$$(m(g^{-1}) - m(h))(m(gh) - m(g)) = 0.$$

Example: Free groups

Consider the free group with 2 generators $\mathbb{F}_2$ acting on it Cayley graph.

$$m = C_1 \mathbb{1}_{\mathcal{W}_a} + C_2 \mathbb{1}_{\mathcal{W}_b} + C_3 \mathbb{1}_{\mathcal{W}_{a^{-1}}} + C_4 \mathbb{1}_{\mathcal{W}_{b^{-1}}}.$$



Groups acting on trees

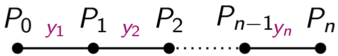
Theorem (Fundamental theorem of Bass-Serre theory)

Let G be a group acting on a tree X without inversion. G can be identified with the fundamental group of a certain graph of groups (G, Y), where $Y = \mathrm{G}\backslash X$, i.e.

$$\mathrm{G} = \pi_1(X, Y, P_0),$$

where P_0 is a vertex of Y.

Let $\mathrm{G} = *_A \mathrm{G}_i$, $i = 0, 1, \cdots, n$. There exists a tree X on which G acts with Y being a series of segments:



Stab(P_i)$= \mathrm{G}_i$ and Stab(y_i)$= A$.

Let $\mathrm{G} = *_A \mathrm{G}_i$, $i = 0, 1, \cdots, n$. For any $g \in \mathrm{G}$ there is a sequence $\mathbf{i} = (i_1, \cdots, i_\ell)$ and a unique reduced word such that

$$g = s_{i_1} \cdots s_{i_\ell} a,$$

where s_{i_j} is a left coset representative of G_{i_j} modulo A.

Theorem (González Pérez-Parcet-X)

Let $\mathrm{G} = *_A \mathrm{G}_i$, $i = 0, 1, \cdots, n$. Then the symbol of the Hilbert transform we defined satisfies the following relation

$$m(g) = C_{s_{i_1}}$$

for any $g = s_{i_1} \cdots s_{i_\ell} a \notin \mathrm{G}_0$.

Thank you!