On the quantum symmetry of distance-transitive graphs

Simon Schmidt

University of Glasgow

April 6, 2021

Quantum automorphism groups of graphs

Consider a finite, simple graph G with adjacency matrix A_G. The *quantum automorphism group* $\mathrm{Qut}(G)$ is the compact matrix quantum group $(C(\mathrm{Qut}(G)), u)$, where

$$C(\mathrm{Qut}(G)) := C^*(u_{ij},\, 1 \leq i,j \leq n \,|\, u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{ki} = 1, uA_G = A_G u).$$

Here $uA_G = A_G u$ is nothing but $\sum_k u_{ik}(A_G)_{kj} = \sum_k (A_G)_{ik} u_{kj}$.

Consider a finite, simple graph G with adjacency matrix A_G. The *quantum automorphism group* $\mathrm{Qut}(G)$ is the compact matrix quantum group $(C(\mathrm{Qut}(G)), u)$, where

$$C(\mathrm{Qut}(G)) := C^*(u_{ij},\, 1 \leq i,j \leq n \,|\, u_{ij} = u_{ij}^* = u_{ij}^2,\, \sum_k u_{ik} = \sum_k u_{ki} = 1,\, uA_G = A_G u).$$

Here $uA_G = A_G u$ is nothing but $\sum_k u_{ik}(A_G)_{kj} = \sum_k (A_G)_{ik} u_{kj}$.

Definition (Banica & Bichon, 2007)

We say that G has *no quantum symmetry* if $C(\mathrm{Qut}(G))$ is commutative, or equivalently $C(\mathrm{Qut}(G)) = C(\mathrm{Aut}(G))$. Otherwise, we say that G *does have quantum symmetry*.

Consider a finite, simple graph G with adjacency matrix A_G. The *quantum automorphism group* $\mathrm{Qut}(G)$ is the compact matrix quantum group $(C(\mathrm{Qut}(G)), u)$, where

$$C(\mathrm{Qut}(G)) := C^*(u_{ij},\, 1 \leq i,j \leq n \,|\, u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{ki} = 1, uA_G = A_G u).$$

Here $uA_G = A_G u$ is nothing but $\sum_k u_{ik}(A_G)_{kj} = \sum_k (A_G)_{ik} u_{kj}$.

Definition (Banica & Bichon, 2007)

We say that G has *no quantum symmetry* if $C(\mathrm{Qut}(G))$ is commutative, or equivalently $C(\mathrm{Qut}(G)) = C(\mathrm{Aut}(G))$. Otherwise, we say that G *does have quantum symmetry*.

Question

When does a graph have quantum symmetry?

Criterion for a graph to have quantum symmetry

Definition

Let V be a finite set and let $\sigma : V \to V$, $\tau : V \to V$ be permutations. We say that σ and τ are *disjoint* if $\sigma(i) \neq i$ implies $\tau(i) = i$ and vice versa.

Criterion for a graph to have quantum symmetry

Definition

Let V be a finite set and let $\sigma : V \to V$, $\tau : V \to V$ be permutations. We say that σ and τ are *disjoint* if $\sigma(i) \neq i$ implies $\tau(i) = i$ and vice versa.

Theorem (S., 2020)

Let G be a graph. If the automorphism group $\mathrm{Aut}(G)$ contains a pair of non-trivial, disjoint automorphisms, then G does have quantum symmetry.

Definition

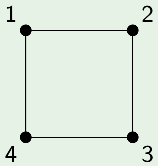
Let V be a finite set and let $\sigma : V \to V$, $\tau : V \to V$ be permutations. We say that σ and τ are *disjoint* if $\sigma(i) \neq i$ implies $\tau(i) = i$ and vice versa.

Theorem (S., 2020)

Let G be a graph. If the automorphism group $\mathrm{Aut}(G)$ contains a pair of non-trivial, disjoint automorphisms, then G does have quantum symmetry.

Example

Consider the quadrangle C_4.



Criterion for a graph to have quantum symmetry

Definition

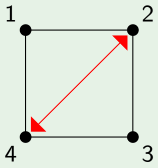
Let V be a finite set and let $\sigma : V \to V$, $\tau : V \to V$ be permutations. We say that σ and τ are *disjoint* if $\sigma(i) \neq i$ implies $\tau(i) = i$ and vice versa.

Theorem (S., 2020)

Let G be a graph. If the automorphism group $\mathrm{Aut}(G)$ contains a pair of non-trivial, disjoint automorphisms, then G does have quantum symmetry.

Example

Consider the quadrangle C_4.



Criterion for a graph to have quantum symmetry

Definition

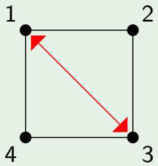
Let V be a finite set and let $\sigma : V \to V$, $\tau : V \to V$ be permutations. We say that σ and τ are *disjoint* if $\sigma(i) \neq i$ implies $\tau(i) = i$ and vice versa.

Theorem (S., 2020)

Let G be a graph. If the automorphism group $\mathrm{Aut}(G)$ contains a pair of non-trivial, disjoint automorphisms, then G does have quantum symmetry.

Example

Consider the quadrangle C_4.



It has disjoint automorphisms $\sigma = (13)$ and $\tau = (24)$. Thus $\mathrm{Qut}(C_4) \neq \mathrm{Aut}(C_4)$.

Graphs having no quantum symmetry

- Strategy: Prove that $C(\mathrm{Qut}(G))$ is commutative by showing that the generators commute

- Strategy: Prove that $C(\mathrm{Qut}(G))$ is commutative by showing that the generators commute
- Recall

$$C(\mathrm{Qut}(G)) = C^*(u_{ij},\, 1 \leq i,j \leq n \,|\, u_{ij} = u_{ij}^* = u_{ij}^2,\, \sum_k u_{ik} = \sum_k u_{ki} = 1, uA_G = A_G u).$$

- Strategy: Prove that $C(\mathrm{Qut}(G))$ is commutative by showing that the generators commute
- Recall

$$C(\mathrm{Qut}(G)) = C^*(u_{ij},\, 1 \leq i,j \leq n \,|\, u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{ki} = 1, uA_G = A_G u).$$

Definition

The distance $d(i,j)$ of vertices i,j is the length of a shortest path connecting i and j.

- In $C(S_n^+)$, we have $uA_G = A_G u$ if and only if $u_{ij}u_{kl} = 0$ for $d(i,k) = 1$ and $d(j,l) \neq 1$ or vice versa

- Strategy: Prove that $C(\mathrm{Qut}(G))$ is commutative by showing that the generators commute
- Recall

$$C(\mathrm{Qut}(G)) = C^*(u_{ij},\, 1 \leq i,j \leq n \,|\, u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{ki} = 1, uA_G = A_G u).$$

Definition

The distance $d(i,j)$ of vertices i,j is the length of a shortest path connecting i and j.

- In $C(S_n^+)$, we have $uA_G = A_G u$ if and only if $u_{ij}u_{kl} = 0$ for $d(i,k) = 1$ and $d(j,l) \neq 1$ or vice versa

Lemma

Let G be a graph and let $(u_{ij})_{1 \leq i,j \leq n}$ be the generators of $C(\mathrm{Qut}(G))$. Then $u_{ij}u_{kl} = 0$ if $d(i,k) \neq d(j,l)$.

Graphs having no quantum symmetry

- Strategy: Prove that $C(\mathrm{Qut}(G))$ is commutative by showing that the generators commute
- Recall

$$C(\mathrm{Qut}(G)) = C^*(u_{ij},\, 1 \leq i,j \leq n \,|\, u_{ij} = u_{ij}^* = u_{ij}^2,\, \sum_k u_{ik} = \sum_k u_{ki} = 1, uA_G = A_G u).$$

Definition

The distance $d(i,j)$ of vertices i,j is the length of a shortest path connecting i and j.

- In $C(S_n^+)$, we have $uA_G = A_G u$ if and only if $u_{ij}u_{kl} = 0$ for $d(i,k) = 1$ and $d(j,l) \neq 1$ or vice versa

Lemma

Let G be a graph and let $(u_{ij})_{1 \leq i,j \leq n}$ be the generators of $C(\mathrm{Qut}(G))$. Then $u_{ij}u_{kl} = 0$ if $d(i,k) \neq d(j,l)$.

Consequence

It remains to show $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l)$.

Definition

Let G be a regular graph. We say that G is *distance-transitive* if for all $(i,k),(j,l) \in V \times V$ with $d(i,k) = d(j,l)$, there is an automorphism $\sigma \in \mathrm{Aut}(G)$ with $\sigma(i) = j, \sigma(k) = l$.

Definition

Let G be a regular graph. We say that G is *distance-transitive* if for all $(i,k),(j,l) \in V \times V$ with $d(i,k) = d(j,l)$, there is an automorphism $\sigma \in \mathrm{Aut}(G)$ with $\sigma(i) = j, \sigma(k) = l$.

Lemma

Let G be a distance-transitive graph. Fix a pair of vertices $a, b \in V$, $d(a,b) = m$. If

$$u_{ia}u_{kb} = u_{kb}u_{ia} \qquad \text{for all } i,k \in V,\ d(i,k) = m,$$

then

$$u_{ij}u_{kl} = u_{kl}u_{ij} \qquad \text{for all } i,j,k,l \in V \text{ with } d(i,k) = d(j,l) = m.$$

Definition

Let G be a regular graph. We say that G is *distance-transitive* if for all $(i,k),(j,l) \in V \times V$ with $d(i,k) = d(j,l)$, there is an automorphism $\sigma \in \mathrm{Aut}(G)$ with $\sigma(i) = j, \sigma(k) = l$.

Lemma

Let G be a distance-transitive graph. Fix a pair of vertices $a, b \in V$, $d(a,b) = m$. If

$$u_{ia}u_{kb} = u_{kb}u_{ia} \qquad \text{for all } i,k \in V,\ d(i,k) = m,$$

then

$$u_{ij}u_{kl} = u_{kl}u_{ij} \qquad \text{for all } i,j,k,l \in V \text{ with } d(i,k) = d(j,l) = m.$$

Consequence

$\#\{$cases for showing commutativity of the generators$\}$ = diameter of G

The intersection array

- We have seen that it is enough to do one computation for every distance
- What about graphs with huge diameter or a series with increasing diameter?

The intersection array

- We have seen that it is enough to do one computation for every distance
- What about graphs with huge diameter or a series with increasing diameter?

Definition

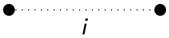
Let G be a distance-transitive graph with diameter d. The *intersection array* of G is a sequence of integers $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$, such that for any two vertices v, w at distance $d(v, w) = i$, there are exactly b_i neighbors of w at distance $i + 1$ to v and exactly c_i neighbors of w at distance $i - 1$ to v.

The intersection array

- We have seen that it is enough to do one computation for every distance
- What about graphs with huge diameter or a series with increasing diameter?

Definition

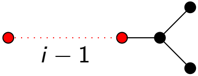
Let G be a distance-transitive graph with diameter d. The *intersection array* of G is a sequence of integers $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$, such that for any two vertices v, w at distance $d(v, w) = i$, there are exactly b_i neighbors of w at distance $i + 1$ to v and exactly c_i neighbors of w at distance $i - 1$ to v.



- We have seen that it is enough to do one computation for every distance
- What about graphs with huge diameter or a series with increasing diameter?

Definition

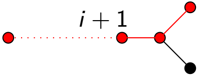
Let G be a distance-transitive graph with diameter d. The *intersection array* of G is a sequence of integers $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$, such that for any two vertices v, w at distance $d(v, w) = i$, there are exactly b_i neighbors of w at distance $i + 1$ to v and exactly c_i neighbors of w at distance $i - 1$ to v.



- We have seen that it is enough to do one computation for every distance
- What about graphs with huge diameter or a series with increasing diameter?

Definition

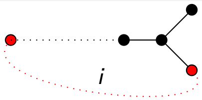
Let G be a distance-transitive graph with diameter d. The *intersection array* of G is a sequence of integers $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$, such that for any two vertices v, w at distance $d(v, w) = i$, there are exactly b_i neighbors of w at distance $i + 1$ to v and exactly c_i neighbors of w at distance $i - 1$ to v.



- We have seen that it is enough to do one computation for every distance
- What about graphs with huge diameter or a series with increasing diameter?

Definition

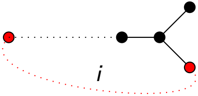
Let G be a distance-transitive graph with diameter d. The *intersection array* of G is a sequence of integers $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$, such that for any two vertices v, w at distance $d(v, w) = i$, there are exactly b_i neighbors of w at distance $i + 1$ to v and exactly c_i neighbors of w at distance $i - 1$ to v.



- We have seen that it is enough to do one computation for every distance
- What about graphs with huge diameter or a series with increasing diameter?

Definition

Let G be a distance-transitive graph with diameter d. The *intersection array* of G is a sequence of integers $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$, such that for any two vertices v, w at distance $d(v, w) = i$, there are exactly b_i neighbors of w at distance $i + 1$ to v and exactly c_i neighbors of w at distance $i - 1$ to v.



Lemma

Let G be distance-transitive with intersection array $\{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}$ and let $(u_{ij})_{1 \leq i,j \leq n}$ be the generators of $C(\mathrm{Qut}(G))$. Assume $b_1 + 1 = b_0$, $c_2 = 1$, $c_m \geq 2$ for some $m > 2$ and $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all vertices i, j, k, l with $d(i, k) = d(j, l) = m - 1$. Then we have $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all i, j, k, l with $d(i, k) = d(j, k) = m$.

Example: The odd graphs

Example

Let $k \geq 2$. The *odd graph* O_k is a graph, where

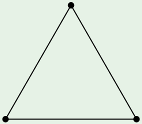
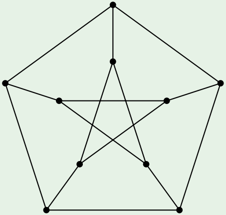
- the vertices correspond to $(k-1)$-subsets of $\{1, \ldots, 2k-1\}$,
- two vertices are connected if and only if the corresponding subsets are disjoint

Example: The odd graphs

Example

Let $k \geq 2$. The *odd graph* O_k is a graph, where

- the vertices correspond to $(k-1)$-subsets of $\{1, \ldots, 2k-1\}$,
- two vertices are connected if and only if the corresponding subsets are disjoint

$O_2=$  , $O_3=$  , ...

Example: The odd graphs

Example

The odd graph O_k has the following properties:

- it is distance-transitive
- it has intersection array

$$\{k, k-1, k-1, \dots, l+1, l+1, l; 1, 1, 2, 2, \dots, l, l\} \text{ for } k = 2l-1,$$
$$\{k, k-1, k-1, \dots, l+1, l+1; 1, 1, 2, 2, \dots, l-1, l-1, l\} \text{ for } k = 2l.$$

Example: The odd graphs

Example

The odd graph O_k has the following properties:

- it is distance-transitive
- it has intersection array

$$\{k, k-1, k-1, \dots, l+1, l+1, l; 1, 1, 2, 2, \dots, l, l\} \text{ for } k = 2l-1,$$
$$\{k, k-1, k-1, \dots, l+1, l+1; 1, 1, 2, 2, \dots, l-1, l-1, l\} \text{ for } k = 2l.$$

The proof that O_k has no quantum symmetry looks as follows:

- First show $u_{ij}u_{kl} = u_{kl}u_{ij}$ for $d(i,k) = d(j,l) \in \{1, 2\}$
- Since we have $b_1 + 1 = b_0$, $c_2 = 1$ and $c_m \geq 2$ for all $m > 2$, we use the previous lemma to finish the proof

Distance-transitive graphs having no quantum symmetry

Theorem (S., 2020)

The following families of distance-transitive graphs do not have quantum symmetry:

- the odd graphs O_k,
- the Johnson graphs $J(n, 2)$, $n \geq 5$,
- the Kneser graphs $K(n, 2)$, $n \geq 5$,
- the Hamming graphs $H(n, 3)$,
- all cubic distance-transitive graphs of order ≥ 10,
- strongly regular graphs with girth five.

Theorem (S., 2020)

The following families of distance-transitive graphs do not have quantum symmetry:

- the odd graphs O_k,
- the Johnson graphs $J(n, 2)$, $n \geq 5$,
- the Kneser graphs $K(n, 2)$, $n \geq 5$,
- the Hamming graphs $H(n, 3)$,
- all cubic distance-transitive graphs of order ≥ 10,
- strongly regular graphs with girth five.

Thank you!

Theorem (S., 2020)

The following families of distance-transitive graphs do not have quantum symmetry:

- the odd graphs O_k,
- the Johnson graphs $J(n, 2)$, $n \geq 5$,
- the Kneser graphs $K(n, 2)$, $n \geq 5$,
- the Hamming graphs $H(n, 3)$,
- all cubic distance-transitive graphs of order ≥ 10,
- strongly regular graphs with girth five.

Thank you!

- S. Schmidt. Quantum automorphisms of folded cube graphs. Ann. Inst. Fourier, Volume 70 (2020) no. 3, pp. 949-970.
- S. Schmidt. On the quantum symmetry of distance-transitive graphs. Adv. Math., 368:107150, 2020.