An algebraist in Operator Algebras

A self-similar perspective

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BMC 2021

Overview

Self-similar actions of groups

Jumping to graph algebras and path spaces

Self-similar actions of groupoids on graph algebras

Self-similar actions of groupoids on k-graphs

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Thinking differently

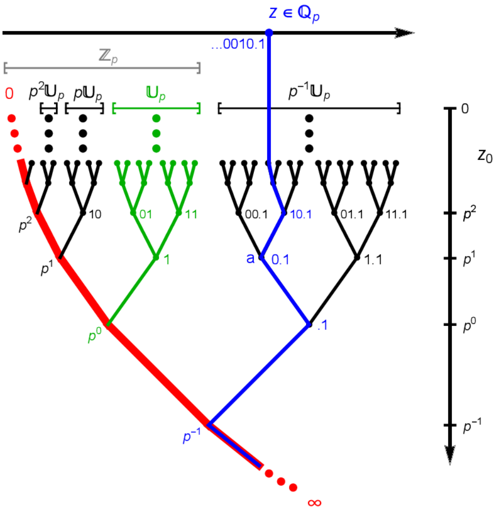

Diagram from *A p-adic version of AdS/CFT*, by S. Gubser, arXiv:1705.00373v1

Automorphisms of graphs help us generate examples of C^*-algebras

Suppose we have a graph Γ with a vertex set Γ^0.

Each vertex $v \in \Gamma^0$ represents a basis element δ_v.

A graph automorphism $\alpha : \Gamma \to \Gamma$ represents a linear map sending $\delta_v \mapsto \delta_{\alpha(v)}$.

The edges between the vertices restrict the operators that can be represented by automorphisms.

We'll look at various types of self-similar actions.

The alphabet X and the tree T_X

Suppose X is a finite set, X^k is the set of k-tuples in X, with $X^0 = \{*\}$, and define $X^* := \bigsqcup_{k\geq 0} X^k = \{\text{finite words in } X\}$.

$T = T_X$ is an infinite homogeneous rooted tree with

- vertex set $T_X^0 = X^* = \{\mu \in X^*\}$
- edge set $T_X^1 = \{\{\mu, \mu x\} \colon \mu \in X^* \text{ and } x \in X\}$
- root the empty word, $*$

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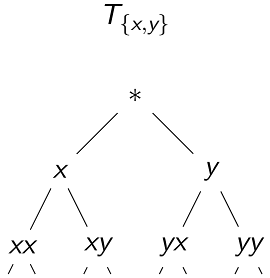

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We label

- edges in T_X with elements of X
- paths and vertices in T_X with elements of X^*.

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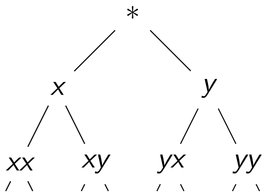
$T_{\{x,y\}}$



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The boundary X^ω of T_X can be identified with semi-infinite words in X starting at $*$, so $X^\omega = \{x_1 x_2 \ldots \ : x_i \in X\}$.

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Automorphisms of $T = T_X$

From a traditional graph-theoretic perspective, an automorphism α of T consists of a family of bijections $\alpha_k \colon X^k \to X^k$ for $k \geq 0$ such that for all $\mu, \nu \in X^*$

$$\{\alpha_k(\mu), \alpha_{k+1}(\nu)\} \in T^1 \quad \Leftrightarrow \quad \{\mu, \nu\} \in T^1.$$

Lemma

Suppose $\alpha \colon T^0 \to T^0$ is a bijection satisfying

$$\alpha(X^k) = X^k \quad \text{for all } k, \text{ and} \quad \alpha(\mu x) \in \alpha(\mu)X \quad \text{for all } \mu \in X^k \text{ and } x \in X. \tag{1}$$

Define $\alpha_k := \alpha|_{X^k}$. Then $\{\alpha_k\}$ is an automorphism α of T. The inverse is also an automorphism of T, and also satisfies (1).

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If $\beta = \{\beta_k\}$ is an automorphism, each $\{\beta_k(\mu), \beta_{k+1}(\mu x)\} \in T^1$, hence $\beta_{k+1}(\mu x) \in \beta_k(\mu)X$. So (1) provides an alternative, ostensibly weaker, characterisation of automorphisms.

Action of a group on T_X

A group G acts (by automorphisms) on T_X if it preserves adjacency (and hence depth).

Consider actions on X^* induced by an action on T_X.

In particular, the action of $g \in G$ can not split a path apart, but its action on an edge labelled $x \in X$ may differ depending on the level.

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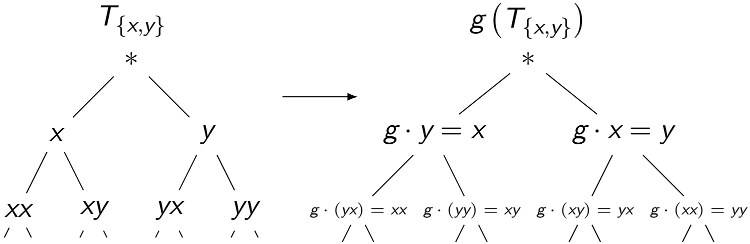
So, in general, $\quad g \cdot (vw) \neq (g \cdot v)(g \cdot w) \quad$ for $g \in G$, $v, w \in X^*$.



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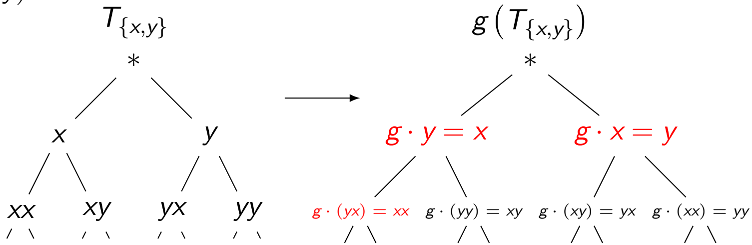
Here, $g \cdot (yx) = xx \neq (g \cdot x)(g \cdot y)$



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Definition of a self-similar action

A self-similar action is a pair (G, X) consisting of a group G and a finite alphabet X with a faithful action of G on X^* satisfying $g \cdot * = *$ and

$$\text{for all } (g, x) \in G \times X, \text{ there exist } (h, y) \in G \times X \text{ such that}$$

$$g \cdot (xw) = y(h \cdot w) \quad \text{for all } w \in X^*$$

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It follows that

$$\text{for all } g \in G, v \in X^*, \text{ there exists a unique } h \in G \text{ such that}$$

$$g \cdot (vw) = (g \cdot v)(h \cdot w) \quad \text{for all } w \in X^*$$

Call this $h \in G$ the restriction of g at v and write $h = g|_v$.

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An example - the odometer

Let $G = \mathbb{Z} = \langle a \rangle$ and $X = \{0, 1\}$.

Define an action of $\mathbb{Z}$ on X^* recursively by

$$\begin{aligned} a \cdot (0w) &= 1w \\ a \cdot (1w) &= 0(a \cdot w) \end{aligned}$$

This corresponds to the diadic adding machine;
it coincides with the rule of adding one to a diadic integer
(with place value increasing towards the right).

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Another example - the Basilica group

Let $X = \{0, 1\}$ and

$$G = \langle a, b \colon \sigma^n([a, a^b]) \text{ for all } n \in \mathbb{N} \rangle$$

where σ is the substitution $\sigma(b) = a$ and $\sigma(a) = b^2$.

Define an action of G on X^* recursively by

$$\begin{aligned} a \cdot (0w) &= 1(b \cdot w) & \qquad b \cdot (0w) &= 0(a \cdot w) \\ a \cdot (1w) &= 0w & \qquad b \cdot (1w) &= 1w \end{aligned}$$

The Basilica group is an iterated monodromy group with many interesting properties, including being amenable.

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The nucleus

A nucleus of a self-similar action (G, X) is a minimal set $\mathcal{N} \subseteq G$ satisfying the property

for each $g \in G$, there exists $N \in \mathbb{N}$ such that
$$g|_v \in \mathcal{N} \text{ for all words } v \in X^n \text{ with } n \geq N.$$

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A self-similar action is contracting if it has a finite nucleus.

For a contracting self-similar action (G, X), the nucleus is unique and equal to

$$\mathcal{N} = \bigcup_{g \in G} \bigcap_{n \geq 0} \{g|_v \colon v \in X^*, |v| \geq n\}$$

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The bimodule and algebras

Given a self-similar action (G, X), let $C^*(G)$ be the full group C^*-algebra of G and define

$$M = M_{(G,X)} = \bigoplus_{x \in X} C^*(G).$$

M can be given the structure of a free right Hilbert $C^*(G)$-module and we can build a faithful, nondegenerate representation $U \colon G \to \mathcal{UL}(M)$.

We can build Cuntz-Pimsner algebras $\mathcal{O}(G, X)$ (Nekrashevych) and Toeplitz algebras $\mathcal{T}(G, X)$ (Laca, R., Raeburn, Whittaker) and we can explicitly calculate KMS states (LRRW).

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If (G, X) is contracting with nucleus $\{e\}$ then $\mathcal{O}(G, X) = \mathcal{O}_{|X|}$.

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If (G, X) is contracting with nucleus $\{e\}$ then $\mathcal{O}(G, X) = \mathcal{O}_{|X|}$.

There's a combinatorial way of calculating the nucleus using the *Moore diagram*;

this can be used to calculate the KMS states.

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Example: basilica group

$$G = \langle a, b \colon \sigma^n([a, a^b]) \text{ for all } n \in \mathbb{N} \rangle,$$

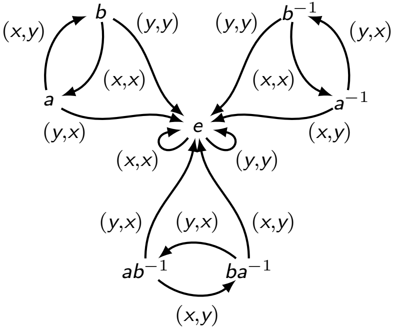
where σ is the substitution $\sigma(b) = a$ and $\sigma(a) = b^2$, with a self-similar action (G, X) where $X = \{0, 1\}$ satisfying

$$a \cdot (0w) = 1(b \cdot w) \qquad b \cdot (0w) = 0(a \cdot w)$$
$$a \cdot (1w) = 0w \qquad b \cdot (1w) = 1w$$

Proposition

The basilica group action (G, X) is contracting, with nucleus

$$\mathcal{N} = \{e, a, b, a^{-1}, b^{-1}, ab^{-1}, ba^{-1}\}.$$



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Example: basilica group

The critical value for KMS_β states is $\beta_c = \ln|X| = \ln 2$.

Proposition

The system $(\mathcal{O}(G,X),\sigma)$ *has a unique* $KMS_{\ln 2}$ *state, which is given on the nucleus* $\mathcal{N} = \{e, a, b, a^{-1}, b^{-1}, ab^{-1}, ba^{-1}\}$ *by*

$$\phi(u_g) = \begin{cases} 1 & \text{for } g = e \\ \frac{1}{2} & \text{for } g = b, b^{-1} \\ 0 & \text{for } g = a, a^{-1}, ab^{-1}, ba^{-1}. \end{cases}$$

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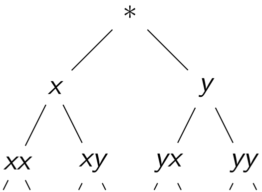
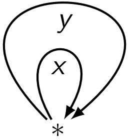
Path space interpretation

$T_{\{x,y\}}$



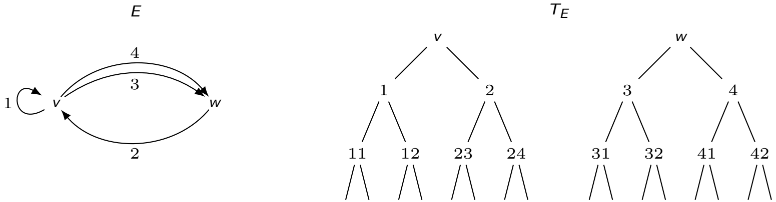
The tree $T_{\{x,y\}}$ represents the path space of the graph



More generally, T_X represents the path space of a bouquet of $|X|$ loops.

Path spaces of graph algebras: from trees to forests

The path space of a finite directed graph E is a forest T_E of rooted trees.



Problems arise:

- The trees in the forest are not necessarily homogeneous.
- Restrictions need not be uniquely determined.
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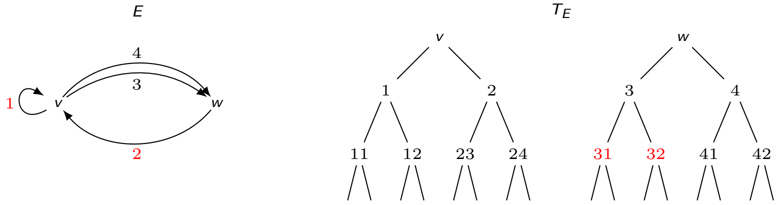

Problems arise:

- The trees in the forest are not necessarily homogeneous.
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- Automorphisms of T_E need not be graph automorphisms of E.

In particular, in general the source map is not equivariant $s(g \cdot e) \neq g \cdot s(e)$ (eg swapping 31 and 32)

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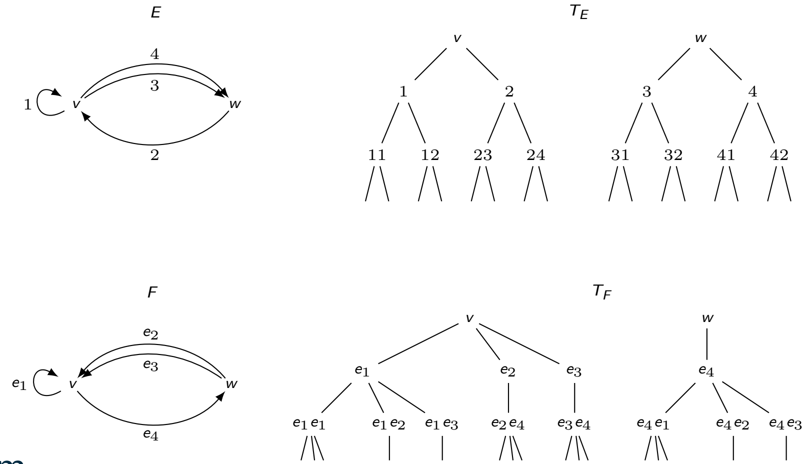
Small changes make big differences



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Path spaces of finite directed graphs, E

Generalise: replace X by edges E^1 in a finite directed graph E.

Suppose $E = (E^0, E^1, r, s)$ is a directed graph with vertex set E^0, edge set E^1, and range and source maps $r, s\colon E^1 \to E^0$. Write

$$E^k = \{\mu = \mu_1 \cdots \mu_k \colon \mu_i \in E^1, s(\mu_i) = r(\mu_{i+1})\}$$

for the set of paths of length k in E, E^0 for the set of vertices, and define $E^* := \bigsqcup_{k\geq 0} E^k$.

We recover the previous work by taking E to be the graph $(\{*\}, X, r, s)$ in which $r(x) = r(y) = s(x) = s(y) = *$ for all $x, y \in X = E^1$ and $E^* = X^*$.

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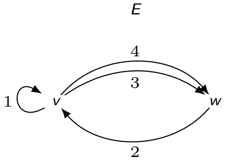
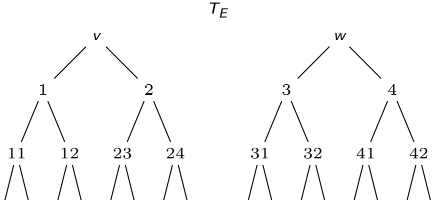
Path space T_E of finite directed graph E

The analogue of the tree T_X is the (undirected) graph T_E with vertex set $T^0 = E^*$ and edge set

$$T^1 = \left\{ \{\mu, \mu e\} : \mu \in E^*, e \in E^1, \text{ and } s(\mu) = r(e) \right\}.$$

The subgraph $vE^* = \{\mu \in E^* : r(\mu) = v\}$ is a rooted tree with root $v \in E^0$, and $T_E = \bigsqcup_{v \in E^0} vE^*$ is a disjoint union of trees, or *forest*.





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Partial isomorphisms

Restrictions become problematic in this context; knowing an action on one tree in the forest doesn't constrain the action on other trees.

Suppose $E = (E^0, E^1, r, s)$ is a directed graph.

A partial isomorphism of T_E consists of two vertices $v, w \in E^0$ and a bijection $g : vE^* \to wE^*$ such that

$$g|_{vE^k} \text{ is a bijection onto } wE^k \text{ for } k \in \mathbb{N}, \text{ and}$$
$$g(\mu e) \in g(\mu)E^1 \text{ for all } \mu e \in vE^*.$$

For $v \in E^0$, we write $\mathrm{id}_v : vE^* \to vE^*$ for the partial isomorphism given by $\mathrm{id}_v(\mu) = \mu$ for all $\mu \in vE^*$.

Denote the set of all partial isomorphisms of T_E by $\mathrm{PIso}(E^*)$.

Define domain and codomain maps $d, c\colon \mathrm{PIso}(E^*) \to E^0$ so that $g : d(g)E^* \to c(g)E^*$.

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Groupoids

A groupoid consists of

- a set G of morphisms,
- a set $G^0 \subseteq G$ of objects (the unit space of the groupoid),
- two functions $c, d : G \to G^0$, and
- a partially defined product $(g, h) \mapsto gh$ from

$$G^2 := \{(g, h) : d(g) = c(h)\} \text{ to } G$$

such that (G, G^0, c, d) is a category and such that each $g \in G$ has an inverse g^{-1}.

We write G to denote the groupoid. If $|G^0| = 1$, then G is a group.

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$(\mathsf{PIso}(E^*), E^0, c, d)$ is a groupoid

Proposition

Suppose that $E = (E^0, E^1, r, s)$ is a directed graph with associated forest T_E.

Then $(\mathsf{PIso}(E^*), E^0, c, d)$ is a groupoid in which:

- *the product is given by composition of functions,*
- *the identity isomorphism at $v \in E^0$ is $\mathrm{id}_v : vE^* \to vE^*$, and*
- *the inverse of $g \in \mathsf{PIso}(E^*)$ is the inverse of the function $g : d(g)E^* \to c(g)E^*$.*

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Groupoid action

Suppose that E is a directed graph and G is a groupoid with unit space E^0.

An action of G on the path space E^* is a (unit-preserving) groupoid homomorphism $\phi : G \to \mathsf{PIso}(E^*)$.

The action is faithful if ϕ is one-to-one.

If the homomorphism is fixed, we usually write $g \cdot \mu$ for $\phi_g(\mu)$.
This applies in particular when G arises as a subgroupoid of $\mathsf{PIso}(E^*)$.

Self-similar groupoid action (G, E)

Definition

Suppose $E = (E^0, E^1, r, s)$ is a directed graph and G is a groupoid with unit space E^0 which acts faithfully on T_E.

The action is self-similar if for every $g \in G$ and $e \in d(g)E^1$, there exists $h \in G$ such that

$$g \cdot (e\mu) = (g \cdot e)(h \cdot \mu) \text{ for all } \mu \in s(e)E^*. \tag{2}$$

Since the action is faithful, there is then exactly one such $h \in G$, and we write $g|_e := h$. Say (G, E) is a self-similar groupoid action.

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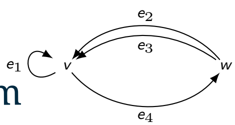

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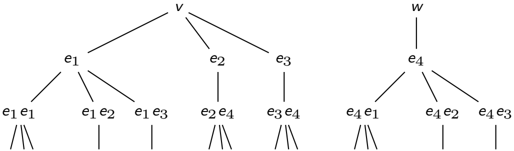

Consequences of self-similar groupoid definition

Lemma

Suppose $E = (E^0, E^1, r, s)$ is a directed graph and G is a groupoid with unit space E^0 acting self-similarly on T_E.

Then for $g, h \in G$ with $d(h) = c(g)$ and $e \in d(g)E^1$, we have

- $d(g|_e) = s(e)$ *and* $c(g|_e) = s(g \cdot e)$,
- $r(g \cdot e) = g \cdot r(e)$ *and* $s(g \cdot e) = g|_e \cdot s(e)$,
- *if* $g = \mathrm{id}_{r(e)}$, *then* $g|_e = \mathrm{id}_{s(e)}$, *and*
- $(hg)|_e = (h|_{g \cdot e})(g|_e)$.

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- *if* $g = \mathrm{id}_{r(e)}$, *then* $g|_e = \mathrm{id}_{s(e)}$, *and*
- $(hg)|_e = (h|_{g \cdot e})(g|_e)$.

Note that in general $s(g \cdot e) \neq g \cdot s(e)$, ie the source map is not G-equivariant. Indeed, $g \cdot s(e)$ will often not make sense: g maps $d(g)E^*$ onto $c(g)E^*$, and $s(e)$ is not in $d(g)E^*$ unless $s(e) = d(g)$.

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Constructing self-similar groupoid actions

We use automata to construct self-similar groupoid actions.

An automaton over $E = (E^0, E^1, r_E, s_E)$ is

- a finite set A containing E^0, with
- functions $r_A, s_A \colon A \to E^0$ such that $r_A(v) = v = s_A(v)$ if $v \in E^0 \subset A$, and
- a function

$$\begin{array}{ccc} A \;{}_{s_A}\!\times_{r_E} E^1 & \to & E^1 \;{}_{s_E}\!\times_{r_A} A \\ (a, e) & \mapsto & (a \cdot e, a|_e) \end{array}$$

such that:

(A1) for every $a \in A$, $e \mapsto a \cdot e$ is a bijection $s_A(a)E^1 \to r_A(a)E^1$;
(A2) $s_A(a|_e) = s_E(e)$ for all $(a, e) \in A \;{}_{s_A}\!\times_{r_E} E^1$;
(A3) $r_E(e) \cdot e = e$ and $r_E(e)|_e = s_E(e)$ for all $e \in E^1$.

We extend restriction to paths by defining

$$a|_\mu = (\cdots((a|_{\mu_1})|_{\mu_2})|_{\mu_3}\cdots)|_{\mu_k}.$$

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Constructing self-similar groupoid actions from directed graphs

We use automata over E to construct subgroupoids of $\mathsf{PIso}(E^*)$.

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Suppose we have an automaton A over a directed graph E.
For each $a \in A$, we construct a partial isomorphism f_a of $s(a)E^*$ onto $r(a)E^*$ so that $d(f_a) = s(a)$ and $c(f_a) = r(a)$.

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Constructing self-similar groupoid actions from directed graphs

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The action of G_A is faithful because G_A is constructed as a subgroupoid of $\mathsf{PIso}(E^*)$.

It is possible to construct unfaithful actions from some automata.

x

What about k-graphs?

Inspired by the work of Robertson and Steger on $\tilde{A}$-buildings,
Kumjian and Pask defined a k-graph (Λ, d) to be

- a countable small category Λ with range and source maps r, s, and $\Lambda^0 = \mathrm{Obj}(\Lambda)$, together with
- a degree functor $d\colon \Lambda \to \mathbb{N}^k$ satisfying the factorisation property that: for every $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ with $d(\lambda) = m + n$ there are unique elements $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ with $d(\mu = m$ and $d(\nu) = n$.

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(Afsar, Brownlowe, R, Whittaker) A partial isomorphism of Λ consists of vertices $v, w \in \Lambda^0$ and a bijection $g : v\Lambda \to w\Lambda$ satisfying

- for all $p \in \mathbb{N}^k$, the restriction $g|_{v\Lambda^p}$ is a bijection of $v\Lambda^p$ onto $w\Lambda^p$; and
- $g(\lambda e) \in g(\lambda)\Lambda$ for all $\lambda \in v\Lambda$ and edges $e \in s(\lambda)\Lambda$.

We write $\mathrm{PIso}(\Lambda)$ for the set of all partial isomorphisms of Λ; it's a groupoid, units Λ^0.

Self-similar actions of on k-graphs

Let Λ be a k-graph and let G be a groupoid with unit space $G^{(0)} = \Lambda^0$.

An action of G on Λ is a groupoid homomorphism $\varphi : G \to \mathsf{PIso}(\Lambda)$.

Identifying id_v with v for $v \in \Lambda^0$, we see that φ is unit preserving.

We say φ is *faithful* if it is injective.

When it is not ambiguous to do so, we write $g \cdot \mu$ for $\varphi_g(\mu)$.

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A self-similar groupoid action (G, Λ) consists of a k-graph Λ, a groupoid G with unit space Λ^0 and a faithful action of G on Λ such that for every $g \in G$ and edge $e \in \mathsf{dom}(g)\Lambda$, there exists $h \in G$ satisfying

$$g \cdot (e\lambda) = (g \cdot e)(h \cdot \lambda) \text{ for all } \lambda \in s(e)\Lambda. \tag{3}$$

Since the action is faithful, there is a unique h satisfying (3), and we write $g|_e := h$.

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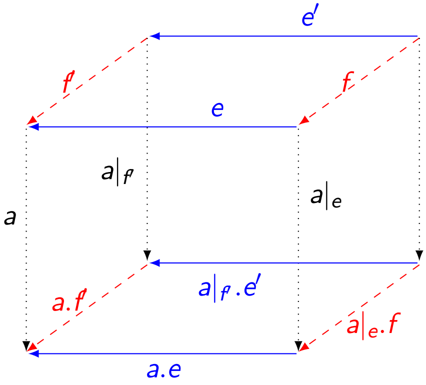
The right framework: coloured graphs, pretty pictures

The right framework in which to develop the k-graph theory appears to be coloured graphs. We define automata associated to coloured graphs, and the relations correspond to multi-dimensional commuting diagrams. So $\lambda := ef \sim f'e' \Rightarrow (a \cdot e)(a|_e \cdot f) \sim (a \cdot f')(a|_{f'} \cdot e')$ corresponds to



$(a|_e)|_f = (a|_{f'})|_{e'}$

What does and doesn't work for k-graphs

We can again construct Toeplitz algebras and calculate KMS states explicitly.

Constraints imposed by product systems limit actions. For example, you can't swap colours.

G-periodicity and tracial states on $C^*(G)$ continue to play a crucial role in calculations.

For k-graphs, G-periodicity appears through the shift map σ and

$$\mathrm{Per}(G,\Lambda) := \left\{ p - q \;:\; \begin{array}{l} p,q \in \mathbb{N}^k, \text{ and there exists } g \in G \text{ such that} \\ \sigma^p(x) = \sigma^q(g \cdot x) \text{ for all } x \in \mathrm{dom}(g)\Lambda^\infty \end{array} \right\}$$

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states X states E x

Questions?

Thank you for your attention.

Questions?

Thank you for your attention.

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States on the Cuntz-Pimsner algebra $\mathcal{O}(G, X)$

Lemma

Let (G, X) be a self-similar action.
If ϕ is a KMS$_\beta$ state on $\mathcal{O}(G, X)$, then $\beta = \ln |X|$.

Lemma

Let (G, X) be a contracting self-similar action with nucleus $\mathcal{N}$.
For each $g \in \mathcal{N} \setminus \{e\}$, let

$$F_g^n = \{\mu \in X^n : g \cdot \mu = \mu \text{ and } g|_\mu = e\}.$$

The sequence $\{|X|^{-n}|F_g^n|\}$ is increasing and converges to a limit c_g satisfying $0 \leq c_g < 1$ and there is a unique KMS$_{\ln |X|}$ state ϕ for $\mathcal{O}(G, X)$ satisfying

$$\phi(u_g) = c_g.$$

States on the Toeplitz algebra $\mathcal{T}(G, X)$

Theorem

Let (G, X) be a self-similar action, $M = \bigoplus_{x \in X} C^*(G)$ and $\sigma : \mathbb{R} \to \operatorname{Aut} \mathcal{T}(G, X)$ satisfy $\sigma_t(s_v u_g s_w^*) = e^{it(|v|-|w|)} s_v u_g s_w^*$ for $v, w \in X^*$ and $g \in G$.

1. *For $\beta < \ln |X|$, there are no KMS_β states.*
2. *For $\beta = \ln |X|$, every $KMS_{\ln |X|}$ state satisfies $\phi_{\ln |X|}(u_g u_h) = \phi_{\ln |X|}(u_h u_g)$ for all $g, h \in G$,*

   $$\phi_{\ln |X|}(s_v u_g s_w^*) = \begin{cases} e^{-(\ln |X|)|v|} \phi_{\ln |X|}(u_g) & \text{if } v = w \\ 0 & \text{otherwise,} \end{cases}$$

   and factors through $\mathcal{O}(G, X)$.
3. *For $\beta > \ln |X|$, the simplex of KMS_β-states of $\mathcal{T}(M)$ is homeomorphic to the simplex of normalized traces on $C^*(G)$ via an explicit construction $\tau \mapsto \psi_{\beta,\tau}$.*

States on the Toeplitz algebra: ψ_{β,τ_e}

Suppose that (G, X) is a self-similar action and $\beta > \ln |X|$.
Suppose τ_e is the trace on $C*(G)$ satisfying

$$\tau_e(\delta_g) = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{otherwise.} \end{cases}$$

For $g \in G$ and $k \geq 0$, we set

$$F_g^k := \{\mu \in X^k : g \cdot \mu = \mu \text{ and } g|_v = e\}.$$

Then there is a KMS$_\beta$ state ψ_{β,τ_e} on $(\mathbb{T}(G, X), \sigma)$ such that

$$\psi_{\beta,\tau_e}(s_v u_g s_w^*) = \begin{cases} e^{-\beta|v|}(1 - |X|e^{-\beta}) \sum_{k=0}^{\infty} e^{-\beta k} |F_g^k| & \text{if } v = w \\ 0 & \text{otherwise.} \end{cases}$$

States on the Toeplitz algebra: ψ_{β,τ_1}

Suppose that (G, X) is a self-similar action and $\beta > \ln|X|$.
Suppose $\tau_1 : C^*(G) \to \mathbb{C}$ is the integrated form of the trivial representation sending $g \mapsto 1$ for all $g \in G$.
For $g \in G$ and $k \geq 0$, we set

$$G_g^k := \{\mu \in X^k : g \cdot \mu = \mu\}.$$

Then there is a KMS$_\beta$ state ψ_{β,τ_1} on $(\mathbb{T}(G, X), \sigma)$ such that

$$\psi_{\beta,\tau_1}(s_v u_g s_w^*) = \begin{cases} e^{-\beta|v|}(1 - |X|e^{-\beta})\sum_{k=0}^{\infty} e^{-\beta k}|G_g^k| & \text{if } v = w \\ 0 & \text{otherwise.} \end{cases}$$

Computing F_g^k and G_g^k: the Moore diagram

Suppose (G, X) is a self-similar action.

A Moore diagram is a directed graph whose vertices are elements of G and edges are labelled by pairs of elements of X.

In a Moore diagram the arrow

$$g \xrightarrow{(x,y)} h$$

means that $g \cdot x = y$ and $g|_x = h$.

We can draw a Moore diagram for any subset S of G that is closed under restriction.

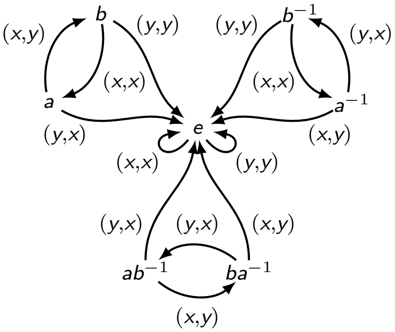
The Moore diagram of the nucleus helps us calculate F_g^k and G_g^k; we look for labels of the form (x, x), called stationary paths.

Computing the nucleus

Proposition

Suppose (G, X) is a self-similar action and S is a subset of G that is closed under restriction. Every vertex in the Moore diagram of S that can be reached from a cycle belongs to the nucleus.

For the basilica group, the minimal Moore diagram we need to consider is



KMS states on $\mathcal{T}(G, E)$

Proposition

Let E be a finite graph with no sources and vertex matrix B, and let $\rho(B)$ be the spectral radius of B.

Suppose that (G, E) is a self-similar groupoid action.

Let $\sigma : \mathbb{R} \to \operatorname{Aut} \mathcal{T}(G, E)$, $\quad \sigma_t(s_\mu u_g s_\nu^*) = e^{it(|\mu|-|\nu|)} s_\mu u_g s_\nu^*$.

- *For $\beta < \ln \rho(B)$, there are no KMS$_\beta$-states for σ.*
- *For $\beta \geq \ln \rho(B)$, a state ϕ is a KMS$_\beta$-state for σ if and only if $\phi \circ i_{C^*(G)}$ is a trace on $C^*(G)$ and*

$$\phi(s_\mu u_g s_\nu^*) = \delta_{\mu,\nu} \delta_{s(\mu),c(g)} \delta_{s(\nu),d(g)} e^{-\beta|\mu|} \phi(u_g)$$

for $g \in S$ and $\mu, \nu \in E^*$.

KMS states on $\mathcal{T}(G,\Lambda)$

Theorem

Let Λ be a finite k-graph with no sources, and (G,Λ) a self-similar groupoid action. For $1 \leq i \leq k$, let B_i be the matrix with entries $B_i(v,w) = |v\Lambda^{e_i}w|$ and let $\rho(B_i)$ be the spectral radius of B_i. Take $r \in (0,\infty)^k$ and let $\sigma : \mathbb{R} \to \operatorname{Aut} \mathbb{T}(G,\Lambda)$ be the dynamics

$$\sigma_t(t_\lambda u_g t_\mu^*) = e^{itr\cdot(d(\lambda)-d(\mu))} t_\lambda u_g t_\mu^*.$$

Suppose that $\beta r_i > \ln \rho(B_i)$ for all $1 \leq i \leq k$.

- *If τ is tracial state on $C^*(G)$, then the series $\sum_{p\in\mathbb{N}^k}\sum_{\lambda\in\Lambda^p} e^{-\beta r\cdot p}\tau(i_{s(\lambda)})$ converges to a positive number $Z(\beta,\tau)$, and there is a KMS$_\beta$-state ϕ_τ of $(\mathbb{T}(G,\Lambda),\sigma)$ such that*

$$\phi_\tau(t_\lambda u_g t_\mu^*) = \delta_{\lambda,\mu} Z(\beta,\tau)^{-1} \sum_{p\geq d(\lambda)} e^{-\beta r\cdot p} \sum_{\{\nu\in s(\lambda)\Lambda^{p-d(\lambda)}:\, g\cdot\nu=\nu\}} \tau\left(i_{g|_\nu}\right).$$

- *The map $\tau \mapsto \phi_\tau$ is an affine isomorphism of the simplex of tracial states of $C^*(G)$ onto the simplex of KMS$_\beta$-states of $(\mathbb{T}(G,\Lambda),\sigma)$.*