An introduction to (quantum) symmetries of (quantum) graphs

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Non-commutative topology

I am under obligation to provide this table:

Spaces	Algebras
Locally compact Hausdorff space	Commutative C*-algebra
Compact	Unital
(Proper) continuous map	$*$-Homomorphism
Cartesian Product	Tensor product

Remember that this relationship is *contravariant*.

How might we deal with (Compact) *groups*?

As the product $G \times G \to G$ and the inverse $G \to G$ and the identity $* \to G$ are continuous maps, we could specify a commutative C^*-algebra A, and $*$-homomorphisms

$$A \to A \otimes A, \qquad A \to A, \qquad A \to \mathbb{C},$$

satisfying appropriate axioms.

What are groups?

Definition

A group is a set G with an associative product $G \times G \to G$ such that:

- There is $e \in G$ with $eg = ge = g$ for each $g \in G$;
- For each $g \in G$ there are $h, k \in G$ with $gh = kg = e$.

So really the identity and inverse are "properties" of the semigroup G, not "structure".

It turns out that we get a (much) more interesting theory if we similarly focus on the product, and think about an extra property.

Compact Quantum groups

Definition (Woronowicz)

A *compact quantum group* is a unital C^*-algebra A together with a unital $*$-homomorphism, the *coproduct*, $\Delta : A \to A \otimes A$ such that:

$$\{(a \otimes 1)\Delta(b) : a, b \in A\}, \quad \{(1 \otimes a)\Delta(b) : a, b \in A\}$$

both have dense linear span in $A \otimes A$.

Theorem

Let (A, Δ) be a compact quantum group with A commutative. There is a compact group G with $A = C(G)$ and $\Delta : C(G) \to C(G) \otimes C(G) = C(G \times G)$ given by

$$\Delta(f)(s, t) = f(st) \qquad (f \in C(G), s, t \in G).$$

Discrete group examples

Let Γ be a discrete group, and form $C_r^*(\Gamma)$ as a concrete C^*-algebra of operators on $\ell^2(\Gamma)$ generated by the translation operators λ_s for $s \in \Gamma$.
There is a $*$-homomorphism

$$\Delta : C_r^*(\Gamma) \to C_r^*(\Gamma) \otimes C_r^*(\Gamma); \quad \lambda_s \mapsto \lambda_s \otimes \lambda_s.$$

Easy to check the density conditions; so $(C_r^*(\Gamma), \Delta)$ is a compact quantum group.
The map representing the unit "should be"

$$\epsilon : C_r^*(\Gamma) \to \mathbb{C}; \quad \lambda_s \mapsto 1.$$

This is only bounded if Γ is amenable.
More generally, we need to look at $C^*(\Gamma)$.

Lots of structure

Let (A, Δ) be a compact quantum group. Then A admits a "Haar state", a state $h : A \to \mathbb{C}$ which is invariant:

$$(h \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes h)\Delta(a) = h(a)1 \qquad (a \in A).$$

The analogue of a (unitary, finite-dimensional) group representation is a *corepresentation*, a unitary matrix $u = (u_{ij}) \in M_n(A)$ with

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj} \qquad (1 \leq i, j \leq n).$$

(Idea: this links Δ with the "dual of matrix multiplication".)
Then corepresentations split into irreducible factors, and we have an entire analogue of Peter–Weyl theory, for example.

Further examples

A *magic unitary* is a matrix $u = (u_{ij}) \in M_n(A)$ (for some unital C^*-algebra A) such that:

- each u_{ij} is a projection: $u_{ij} = u_{ij}^2 = u_{ij}^*$;
- each row and column sums to 1, so $\sum_k u_{kj} = \sum_k u_{ik} = 1$.

These imply in a given row or column, all the projections are mutually orthogonal.

Such a matrix is unitary, as e.g.

$$\sum_k (u^*)_{ik} u_{kj} = \sum_k u_{ki}^* u_{kj} = \sum_k u_{ki} u_{kj} = \delta_{ij} \sum_k u_{ki} = \delta_{ij} 1.$$

Let S_n^+ be the universal unital C^*-algebra generated by a universal magic unitary $(u_{ij})_{i,j=1}^n$.

[I am deliberately confusing the algebra and the "quantum group".]

"Universal" C^*-algebras

"Let S_n^+ be the universal unital C^*-algebra generated by a universal magic unitary $(u_{ij})_{i,j=1}^n$."

- We could take all possible (up to some cardinality) C^*-algebras A with a magic unitary $u = (u_{ij}) \in M_n(A)$ such that the entries u_{ij} generate A. Then take the direct sum.
- Or consider the $*$-algebra with generators (u_{ij}) and relations, and take the enveloping C^*-algebra.
- Notice that $\|u_{ij}\| = 1$ always!
- These constructions are the same.
- Not clear to me what S_n^+ *actually is*!

Abelianisation

$$u_{ij} = u_{ij}^2 = u_{ij}^*, \quad \sum_k u_{ik} = \sum_k u_{kj} = 1.$$

Let $\phi : S_n^+ \to \mathbb{C}$ be a character. Then:

- $\phi(e) \in \{0, 1\}$ for any projection e; and $\phi(1) = 1$;

So the scalar matrix $(\phi(u_{ij}))$ is $0, 1$-valued, and each row and column sums to 1.

- So $(\phi(u_{ij}))$ is a permutation matrix!

So as ϕ varies, we see that we obtain $C(S_n)$, which is hence the abelianisation of S_n^+.

- $C(S_n)$ is hence what you get if we also require each generator u_{ij} to commute.
- S_n^+ is a *liberation* of S_n.

As a quantum group

The elements $v_{ij} := \sum_{k=1}^n u_{ik} \otimes u_{kj} \in S_n^+ \otimes S_n^+$ are also projections, which satisfy the row/column relations. So by universality, there is a $*$-homomorphism

$$\Delta : S_n^+ \to S_n^+ \otimes S_n^+; \quad u_{ij} \mapsto \sum_{k=1}^n u_{ik} \otimes u_{kj}.$$

- Easy to see that Δ is coassociative.
- As the matrix (u_{ij}) is unitary and each u_{ij} is self-adjoint, one can check that the density conditions hold. [Though this is a bit of work.]

So (S_n^+, Δ) is a compact quantum group: the "quantum symmetry group."

But what is it "symmetries" of?

Quantum group (co)actions

An (right) action of a group G on a space/set X is a map

$$X \times G \to X.$$

So we get

$$\alpha : C(X) \to C(X) \otimes C(G),$$

- $(\mathrm{id} \otimes \Delta)\alpha = (\alpha \otimes \mathrm{id})\alpha$ corresponds to $x \cdot st = (x \cdot s) \cdot t$;
- $\mathrm{lin}\{\alpha(b)(1 \otimes a) : a \in C(G), b \in C(X)\}$ is dense in $C(X) \otimes C(G)$ corresponds to $x \cdot e = x$.

Definition (Podleś)

A (right) coaction of a compact quantum group (A, Δ) on a C^*-algebra B is a unital $*$-homomorphism $\alpha : B \to B \otimes A$ with these two conditions.

Coactions on $\mathbb{C}^n$

Fix a compact quantum group (A, Δ).

- The algebra $\mathbb{C}^n$ is spanned by projections $(e_i)_{i=1}^n$.
- So $\alpha : \mathbb{C}^n \to \mathbb{C}^n \otimes A$ is determined by (u_{ij}) in A with

$$\alpha(e_i) = \sum_{j=1}^n e_j \otimes u_{ji}.$$

- α is a $*$-homomorphism $\Leftrightarrow$ each u_{ji} a projection and $u_{ji} u_{jk} = \delta_{ik} u_{ji}$;
- α is unital $\Leftrightarrow$ $\sum_i u_{ji} = 1$;
- α satisfies the coaction equation $\Leftrightarrow$ $\Delta(u_{ji}) = \sum_k u_{jk} \otimes u_{ki}$;
- α satisfies the Podleś density condition $\Leftrightarrow$ $\sum_i u_{ji} = 1$.
- General Theory $\implies$ $\sum_j u_{ji} = 1$. So (u_{ij}) is a magic unitary.

Quantum symmetry group of the space of n points

For $\mathbb{C}^n = C(\{1, 2, \cdots, n\})$,

$$\alpha(e_i) = \sum_{j=1}^{n} e_j \otimes u_{ji},$$

with $u = (u_{ij})$ a magic unitary.

- So there is a quantum group morphism $S_n^+ \to A$.

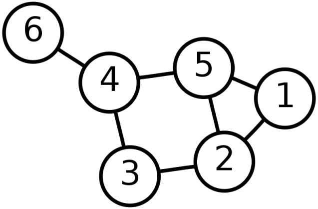
Theorem (Wang)

S_n^+ is the "largest" compact quantum group which acts on $\mathbb{C}^n$ is a "non-degenerate" way.

We think of S_n^+ as the "quantum symmetry group" of $\{1, 2, \cdots, n\}$.

More structure: graphs

Consider a (simple, undirected) graph G on vertex set $V = \{1, 2, \cdots, n\}$. The *adjacency matrix* is $A = A_G$ a $0, 1$-valued matrix with $A_{ij} = 1$ if and only if there is an edge between vertices i and j.



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

So A is symmetric, and if we do not allow loops, then A has 0 on the diagonal.

Automorphisms of graphs

What is a "symmetry" of a graph?

- A permutation of the underlying vertex set;
- which preserves the proper of vertices being neighbours, or not.

Let $T : \mathbb{C}^n \to \mathbb{C}^n$ be the linear map induced by the adjacency matrix. So

$$T(e_i) = \sum_j A_{ji} e_j = \sum_{i \sim j} e_j,$$

where $i \sim j$ when i is adjacent to j.

- Thus an automorphism of a graph is a permutation $\sigma \in S_n$ with

$$TU_\sigma(e_i) = U_\sigma T(e_i) \qquad (1 \leq i \leq n),$$

where $U_\sigma : e_i \mapsto e_{\sigma(i)}$.

(Co)actions on graphs

$$TU_\sigma = U_\sigma T$$

- So $\mathrm{Aut}(G)$ acts in a way which preserves T:

$$\alpha : \mathbb{C}^n \to \mathbb{C}^n \otimes C(\mathrm{Aut}(G)); \quad \alpha T = (T \otimes \mathrm{id})\alpha.$$

Definition (Banica)

The *quantum automorphism group* of G is the maximal compact quantum group $\mathrm{QAut}(G)$ with a coaction satisfying

$$\alpha : \mathbb{C}^n \to \mathbb{C}^n \otimes \mathrm{QAut}(G); \quad \alpha T = (T \otimes \mathrm{id})\alpha.$$

Equivalently, the underlying magic unitary $U = (u_{ij})$ has to commute with the adjacency matrix A.

Some examples

- When $n \leq 3$ we have that $S_n^+ = C(S_n)$.
- For $n \geq 4$ we know that S_n^+ is infinite-dimensional:

$$\begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & 1 \end{pmatrix}$$

- S_4^+ is nuclear; S_n^+ is non-nuclear for $n \geq 5$ [Banica]
- Let $C(S_n^+)$ be the image of S_n^+ acting on the GNS space for the Haar state. Then $C(S_n^+)$ is simple with unique trace, when $n \geq 8$. [Brannan]

Graph Laplacian?

For graphs, the structure of G matters, except when G is the complete graph.

But why look at the *adjacency matrix*?

Consider

$$A^2_{ij} = \sum_k A_{ik} A_{kj} = \big|\{k : i \sim k, j \sim k\}\big|.$$

- In particular, A^2_{ii} is the degree of i. Some work then shows that if $\deg(i) \neq \deg(j)$ then $u_{ij} = 0$ [Fulton].
- Thus if D is the *degree matrix*, $D = \operatorname{diag}(\deg(i))$, then $Du = uD$.
- So also $Lu = uL$ where $L = D - A$ is the graph Laplacian.

Graph Laplacian: converse

Suppose u is a magic unitary with $Lu = uL$. Then

$$(uL)_{ij} = \deg(j)u_{ij} - \sum_{k\sim j} u_{ik}, \quad (Lu)_{ij} = \deg(i)u_{ij} - \sum_{i\sim k} u_{kj}.$$

These agree, so multiply by u_{ij} to get

$$\deg(j)u_{ij} - \sum_{k\sim j} u_{ij}u_{ik} = \deg(i)u_{ij} - \sum_{k\sim i} u_{ij}u_{kj}.$$

As $u_{ij}u_{ik} = \delta_{j,k}u_{ij}$ and $u_{ij}u_{kj} = \delta_{ik}u_{ij}$, we see

- If $i \sim j$ (so $j \sim i$) then $(\deg(j) - 1)u_{ij} = (\deg(i) - 1)u_{ij}$;
- Otherwise $\deg(j)u_{ij} = \deg(i)u_{ij}$;
- In either case, $\deg(i) \neq \deg(j) \implies u_{ij} = 0$.
- So $Du = uD$ and hence $Au = uA$ as $L = D - A$.

[With thanks to Simon Schmidt.]

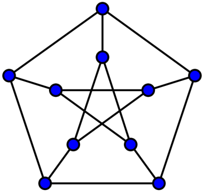
Examples continued

Question

What other matrices / operators associated to G would give the same definition of $\mathrm{QAut}(G)$?

We say that a graph *has quantum symmetry* if $\mathrm{Aut}(G) \neq \mathrm{QAut}(G)$.

- By now, we have many examples.
- For example, the Petersen graph has no quantum symmetry [Schmidt].



- The next talk will say more!