Completely almost periodic elements of group von Neumann algebras

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BMC/BAMC Glasgow 2021
Operator Algebras session
6th April 2021

Let Γ be a locally compact group. For $s \in \Gamma$, define the left translation operator $\lambda_s : L^2(\Gamma) \to L^2(\Gamma)$ by $\lambda_s \xi(t) = \xi(s^{-1}t)$.

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The group von Neumann algebra as a commutant

Let $a \in \mathcal{B}(L^2(\Gamma))$ where Γ is unimodular (e.g. compact, abelian or discrete). Then $a \in \mathrm{VN}(\Gamma)$ iff it commutes with every **right** translation operator on $L^2(\Gamma)$.

Example 1. $\mathrm{VN}(\mathbb{Z})$ is the set of all $a \in \mathcal{B}(\ell^2(\mathbb{Z}))$ satisfying

$$a_{ij} = a_{i+k,j+k} \qquad \text{for all } i, j, k \in \mathbb{Z}.$$

For $\phi \in \mathrm{VN}(\Gamma)_*$ let $\mathrm{ev}_\phi(s) = \phi(\lambda_s)$. Then $\mathrm{ev}_\phi \in C_0(\Gamma)$.

Since $\phi \mapsto \mathrm{ev}_\phi$ is linear and injective we can identify $\mathrm{VN}(\Gamma)_*$ with a linear subspace of $C_0(\Gamma)$; this is the Fourier algebra $\mathrm{A}(\Gamma)$.

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The LCA case

If Γ is loc. compact abelian, with dual group $G = \widehat{\Gamma}$, then the Fourier transform $L^2(\Gamma) \cong L^2(G)$ induces isomorphisms $\mathrm{VN}(\Gamma) \cong L^\infty(G)$ and $\mathrm{A}(\Gamma) \cong L^1(G)$.

Convolution in $L^1(G)$ corresponds to pointwise multiplication in $\mathrm{A}(\Gamma)$

For general Γ: $\mathrm{A}(\Gamma)$ is still an algebra with respect to pointwise multiplication; thus it acts on $\mathrm{VN}(\Gamma) = \mathrm{A}(\Gamma)^*$, via:

$$\langle x \bullet \mathrm{ev}_\phi, \mathrm{ev}_\psi \rangle_{\mathrm{VN}(\Gamma)-\mathrm{A}(\Gamma)} = \langle x, \mathrm{ev}_\phi \cdot \mathrm{ev}_\psi \rangle_{\mathrm{VN}(\Gamma)-\mathrm{A}(\Gamma)}$$

or, by slight abuse of notation, $\langle x \bullet \phi, \psi \rangle = \langle x, \phi \cdot \psi \rangle$.

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Example 2. If $a \in \mathrm{VN}(\mathbb{Z})$ and $\phi \in \mathrm{A}(\mathbb{Z})$ then $(a \bullet \phi)_{ij} = a_{ij}\phi(i-j)$.

Remark

If Γ is discrete and we view elements of $\mathrm{VN}(\Gamma)$ as infinite matrices, then $\mathrm{A}(\Gamma)$ acts on $\mathrm{VN}(\Gamma)$ by (a version of) Schur multiplication.

Since $\mathrm{VN}(\Gamma)_*$ is an algebra acting on $\mathrm{VN}(\Gamma)$, for each $x \in \mathrm{VN}(\Gamma)$ we get an orbit map for this action:

$$L_x : \mathrm{VN}(\Gamma)_* \to \mathrm{VN}(\Gamma) \quad , \quad \phi \mapsto x \bullet \phi$$

Definition (Dunkl–Ramirez (1973))

$$\mathrm{AP}(\widehat{\Gamma}) := \{x \in \mathrm{VN}(\Gamma) \mid L_x : \mathrm{VN}(\Gamma)_* \to \mathrm{VN}(\Gamma) \textit{ is compact}\}$$

This space has been studied by various authors (Granirer, 1974; Lau, 1977; Chou, 1990) in the context of abstract harmonic analysis. But where does it come from?

Let G be a group and let $f \in \ell^\infty(G)$. We say that f is almost periodic if the set of left translates of f is a totally bounded subset of $\ell^\infty(G)$.

Example 3. For any $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and any $c_1, \ldots, c_n \in \mathbb{C}$, the function

$$f(t) = c_1 e^{i\alpha_1 t} + \cdots + c_n e^{i\alpha_n t}$$

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When G is locally compact, the a.p. elements of $L^\infty(G)$ admit a simple operator-theoretic characterization.

Recall: $L^1(G)$ is an algebra (with convolution as the product); so it has a natural right action on $L^1(G)^* = L^\infty(G)$, which we denote by $\bullet$.

Theorem (various sources, 1930s–1970s)

Let $f \in L^\infty(G)$. TFAE:

1. *f is (equal a.e. to) a bounded a.p. function;*
2. *$f \in C_b(G)$ and $\{f \bullet \mu \mid \mu \in \mathrm{Prob}(G)\}$ is totally bounded;*
3. *$L_f : L^1(G) \to L^\infty(G)$, $\mu \mapsto f \bullet \mu$, is a compact operator.*

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The proof uses the existence of a b.a.i. in $L^1(G)$ and various topologies on $L^\infty(G)$ and $M(G)$. The full details are rather technical.

One part is easy: if $L_f : L^1(G) \to L^\infty(G))$ is compact, then $f \in C_b(G)$.

Idea of the proof

Take a b.a.i. (μ_i) in $L^1(G)$; then $f \bullet \mu_i \in C_b(G)$ and $f = \mathrm{w}^*\mathrm{lim}_i\, f \bullet \mu_i$. But some subnet of $(f \bullet \mu_i)$ converges in norm.

A reminder of the D–R definition

$$\mathrm{AP}(\widehat{\Gamma}) := \{x \in \mathrm{VN}(\Gamma) \mid L_x : \mathrm{VN}(\Gamma)_* \to \mathrm{VN}(\Gamma) \text{ is compact}\}$$

If Γ is **abelian** this is the space of continuous a.p. functions on $\widehat{\Gamma}$.

Note that if $x = \lambda_s$ then $L_x(\phi) = \phi(x)x$. Hence

$$\mathrm{C}^*_\delta(\Gamma) := \overline{\mathrm{lin}}\{\lambda_s \mid s \in \Gamma\} \subseteq \mathrm{AP}(\widehat{\Gamma})$$

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When Γ is **discrete** $\mathrm{C}^*_\delta(\Gamma) = \mathrm{C}^*_r(\Gamma)$. Granirer (1974) observed that if Γ is discrete and **amenable**, then $\mathrm{AP}(\widehat{\Gamma}) = \mathrm{C}^*_r(\Gamma)$.

Idea of the proof

Take a b.a.i. (ϕ_i) in $\mathrm{A}(\Gamma)$; then $x \bullet \phi_i \in \mathrm{C}^*_r(\Gamma)$ and $x = \mathrm{w}^*\lim_i x \bullet \phi_i$.
But some subnet of $(x \bullet \phi_i)$ converges in norm.

One serious problem is the following gap in our knowledge:

Question.

Is $\mathrm{AP}(\widehat{\Gamma})$ is a C^*-subalgebra of $\mathrm{VN}(\Gamma)$?

No counterexamples are known, but the question is open even for $\Gamma = \mathbb{F}_2$ or $\Gamma = \mathrm{SU}(2)$!

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No counterexamples are known, but the question is open even for $\Gamma = \mathbb{F}_2$ or $\Gamma = \mathrm{SU}(2)$! The best result to date is:

Theorem (Chou, 1990)

Let Γ be amenable and have an open abelian subgroup. Then $\mathrm{AP}(\Gamma) = \mathrm{C}^*_\delta(\widehat{\Gamma})$; in particular it is a C^*-algebra.

By results of Chou (1990) and Rindler (1992) there are compact (profinite) groups Γ such that the inclusion $\mathrm{C}^*_\delta(\Gamma) \subseteq \mathrm{AP}(\widehat{\Gamma})$ is proper.

Definition (Oikhberg, PhD thesis, 1998)

Given operator spaces V and W and $f \in \mathcal{CB}(V, W)$, we say f is Gelfand completely compact (g.c.c.) if:
for all $\varepsilon > 0$ there exists a closed subspace $E \subset V$ with $\dim(V/E) < \infty$, such that

$$\|E \hookrightarrow V \to W\|_{cb} < \varepsilon.$$

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If we replace the cb-norm with the "usual norm" of $\mathcal{B}(V,W)$ we recover the usual class of compact operators. In particular:

- every g.c.c. map $V \to W$ is compact in the usual sense;
- if $W = L^\infty(G)$ then every compact map $V \to W$ is g.c.c.

Definition (Reformulation of Runde, 2011)

$$\mathrm{CAP}(\widehat{\Gamma}) := \{x \in \mathrm{VN}(\Gamma) \mid L_x : \mathrm{VN}(\Gamma)_* \to \mathrm{VN}(\Gamma) \text{ is } \textbf{g.c.c.}\}$$

Remarks for the specialists

1. Runde (2011) gives a definition that works for all Hopf–von Neumann algebras (M, Δ); when applied to $(L^\infty(G), \widehat{\Delta})$ it recovers the classical space $\mathrm{AP}(G)$.
2. Our formulation is cheating slightly by exploiting the fact that $(\mathrm{VN}(\Gamma), \Delta)$ is co-commutative; for general (M, Δ) we need to add a corresponding condition on $R_x : \mathsf{M}_* \to \mathsf{M}$.
3. Runde's original definition used a different version of operator-space compactness, called (Saar) complete compactness.

There is a normal, unital $*$-homomorphism $\Delta : \mathrm{VN}(\Gamma) \to \mathrm{VN}(\Gamma \times \Gamma)$ satisfying

$$\Delta\lambda_s = \lambda_s \otimes \lambda_s \qquad (s \in \Gamma)$$

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The product on $\mathrm{VN}(\Gamma)_* = \mathrm{A}(\Gamma)$ can be recovered via:

$$(\phi \cdot \psi)(x) = (\phi \otimes \psi)\Delta x \qquad (\phi, \psi \in \mathrm{VN}(\Gamma)_* \, ; x \in \mathrm{VN}(\Gamma))$$

Hence the orbit map $L_x : \mathrm{VN}(\Gamma)_* \to \mathrm{VN}(\Gamma)$ is given by:

$$L_x(\phi) = x \bullet \phi = (\phi \otimes \iota)\Delta x$$

For von Neumann algebras M and N we have $\mathcal{CB}(\mathsf{M}_*, \mathsf{N}) = \mathsf{M}\overline{\otimes}\mathsf{N}$:

$$w \in \mathsf{M}\overline{\otimes}\mathsf{N} \text{ corresponds to } \phi \mapsto (\phi \otimes \iota)(w)$$

Moreover: if N is **injective**, the g.c.c. maps correspond to $\mathsf{M} \otimes_{\min} \mathsf{N}$.

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Moreover: if N is **injective**, the g.c.c. maps correspond to $\mathsf{M} \otimes_{\min} \mathsf{N}$.

Theorem (Runde, 2011)

Suppose that $\mathrm{VN}(\Gamma)$ is injective (e.g. Γ amenable or connected). Then

$$\mathrm{CAP}(\widehat{\Gamma}) = \{x \in \mathrm{VN}(\Gamma) \mid \Delta x \in \mathrm{VN}(\Gamma) \otimes_{\min} \mathrm{VN}(\Gamma)\}$$

In particular, since Δ is a unital $*$-homomorphism, $\mathrm{CAP}(\widehat{\Gamma})$ is a unital C^*-subalgebra of $\mathrm{VN}(\Gamma)$.

Lemma

Let V, W be operator spaces, let $f \in \mathcal{CB}(V, W)$, and let $\iota : W \to N$ be a complete isometry where N is injective. Then

$$f : V \to W \text{ is g.c.c.} \iff \iota f : V \to N \text{ is g.c.c.} \iff \iota f \in V^* \otimes_{\min} N$$

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Theorem (C., in preparation)

$$\mathrm{CAP}(\widehat{\Gamma}) = \{x \in \mathrm{VN}(\Gamma) \mid \Delta x \in \mathrm{VN}(\Gamma) \otimes_{\min} \mathcal{B}(\ell^2(\Gamma))\}$$

In particular $\mathrm{CAP}(\widehat{\Gamma})$ is a unital C^*-subalgebra of $\mathrm{VN}(\Gamma)$.

Ongoing project

Revisit the results of Chou (1990) using $\mathrm{CAP}(\widehat{\Gamma})$ instead of $\mathrm{AP}(\widehat{\Gamma})$.

For the rest of this talk we focus on the case where Γ is discrete. In this case we have $\mathrm{C}^*_r(\Gamma) \equiv \mathrm{C}^*_\delta(\Gamma) \subseteq \mathrm{CAP}(\widehat{\Gamma})$ and the question is: do we always have equality?

For the rest of this talk we focus on the case where Γ is discrete. In this case we have $\mathrm{C}^*_r(\Gamma) \equiv \mathrm{C}^*_\delta(\Gamma) \subseteq \mathrm{CAP}(\widehat{\Gamma})$ and the question is: do we always have equality?

We say $a \in \mathcal{B}(\ell^2(\Gamma))$ has finite bandwidth if there is a finite subset $F \subset \Gamma$ such that $a_{s,t} \neq 0 \implies st^{-1} \in F$. The norm-closure of the set of finite-bandwidth operators is denoted by $\mathrm{UC}^*_r(\Gamma)$.

Lemma

Let $\diamond$ denote the Schur product of operators on $\ell^2(\Gamma)$. If $a \in \mathrm{VN}(\Gamma)$ and $b \in \mathcal{B}(\ell^2(\Gamma))$ then $a \diamond b \in \mathrm{UC}^*_r(\Gamma)$.

$$\mathrm{CAP}(\widehat{\Gamma}) = \{x \in \mathrm{VN}(\Gamma) \mid \Delta x \in \mathrm{VN}(\Gamma) \otimes_{\min} \mathcal{B}(\ell^2(\Gamma))\}.$$

Using the description of Δ in terms of the fundamental unitary for $\mathrm{VN}(\Gamma)$, we can construct $\nabla : \mathcal{B}(\ell^2(\Gamma \times \Gamma)) \to \mathcal{B}(\Gamma)$ such that

- $\nabla\Delta(x) = x$ for all $x \in \mathrm{VN}(\Gamma)$;
- $\nabla(a \otimes b) = a \diamond b$ for all $a, b \in \mathcal{B}(\ell^2(\Gamma))$.

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Theorem (C.)

Let $x \in \mathcal{B}(\ell^2(\Gamma))$. TFAE:

1. $x \in \mathrm{CAP}(\widehat{\Gamma})$
2. $x \in \mathrm{VN}(\Gamma) \cap \mathrm{UC}_r^*(\Gamma)$
3. $x \in \mathrm{VN}(\Gamma)$ *and* $\Delta x \in \mathrm{C}_r^*(\Gamma) \otimes_{\min} \mathrm{UC}_r^*(\Gamma)$.

Discrete groups for which $\mathrm{C}_r^*(\Gamma) = \mathrm{VN}(\Gamma) \cap \mathrm{UC}_r^*(\Gamma)$ are said to have the invariant translation approximation property (ITAP).

No examples are known to fail ITAP; the case of $\mathrm{SL}_3(\mathbb{Z})$ remains open!

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No examples are known to fail ITAP; the case of $\mathrm{SL}_3(\mathbb{Z})$ remains open!

Every discrete group with the approximation property has the ITAP (Zacharias, 2006). Hence if Γ is such a group, we have $\mathrm{CAP}(\widehat{\Gamma}) = \mathrm{C}_r^*(\Gamma)$. This includes: free groups; $\mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$.

On the other hand: if Γ does not have ITAP, then $\mathrm{C}_r^*(\Gamma) \subsetneq \mathrm{CAP}(\widehat{\Gamma})$ and therefore $\mathrm{C}_r^*(\Gamma) \subsetneq \mathrm{AP}(\widehat{\Gamma})$.