Dynamic asymptotic dimension

Applications to Homology and K-theory

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joint work with Clément Dell'Aiera, James Gabe, and Rufus Willett

Groupoids and examples

Let G be a locally compact Hausdorff *groupoid*, i.e. we have a partially defined continuous multiplication map

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and a continuous inverse map

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We will assume that G is *ample*, which means that the range map $r : G \to G$ is a local homeomorphism (i.e. G is étale) and G^0 is totally disconnected.

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$\Gamma \ltimes X$ is principal iff $\Gamma \curvearrowright X$ is free. The other two examples are always principal.

Dynamic asymptotic dimension

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Dimension 0
The groupoids with dimension zero are just those which can be written as an increasing union of compact open subgroupoids (AF groupoids).

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Theorem (GWY '17)
Let G be a principal ample groupoid. Then

$$\dim_{\mathrm{nuc}}(C_r^*(G)) \leq \mathrm{dad}(G)$$

Groupoid homology

Crainic & Moerdijk introduced a homology theory for étale groupoids generalizing group homology. For constant $\mathbb{Z}$ coefficients the construction can be done in an elementary way as follows: We have canonical maps

$$d_i : G^{(n)} \to G^{(n-1)}$$

$$d_i(g_1, \ldots, g_n) = \begin{cases} (g_2, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & 1 \leq i \leq n-1 \\ (g_1, \ldots, g_{n-1}) & i = n \end{cases}$$

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Each d_i induces a map $(d_i)_* : C_c(G^{(n)}, \mathbb{Z}) \to C_c(G^{(n-1)}, \mathbb{Z})$ by

$$(d_i)_*(f)(g_1, \ldots, g_{n-1}) = \sum_{d_i(h_1, \ldots, h_n) = (g_1, \ldots, g_{n-1})} f(h_1, \ldots, h_n)$$

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If we let

$$\delta_n = \sum_{i=0}^{n} (-1)^i (d_i)_*$$

we obtain a chain complex

$$\cdots \to C_c(G^{(2)}, \mathbb{Z}) \xrightarrow{\delta_2} C_c(G^{(1)}, \mathbb{Z}) \xrightarrow{\delta_1} C_c(G^0, \mathbb{Z}) \to 0$$

Groupoid homology and Matui's HK conjecture

The groupoid homology is now just the homology of this chain complex:

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HK conjecture (Matui)
Let G be an essentially principal, minimal ample groupoid, then

$$K_0(C_r^*(G)) \cong \bigoplus_{n=0}^{\infty} H_{2n}(G) \text{ and } K_1(C_r^*(G)) \cong \bigoplus_{n=0}^{\infty} H_{2n+1}(G)$$

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- The conjecture in this form has been disproved by Scarparo.
- Yet, it remains an interesting question which groupoids G DO satisfy the conclusion of the conjecture.

Main results

Theorem
Let G be a principal, σ-compact, ample groupoid with compact unit space and dynamic asymptotic dimension at most d. Then

$$H_n(G) = 0 \text{ for all } n > d, \text{ and } H_d(G) \text{ is free abelian.}$$

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Combining this with a spectral sequence result

$$H_p(G, K_q(C(G^0))) \Rightarrow K_{p+q}(C_r^*(G))$$

recently developed by Proietti-Yamashita (more about that later in Makoto's talk!) we obtain

Corollary (Matui's conjecture up to dimension 2)
Let G be a σ-compact, principal ample groupoid with compact unit space and $\mathrm{dad}(G) \leq 2$, then

$$K_0(C_r^*(G)) \cong H_0(G) \oplus H_2(G), \text{ and } K_1(C_r^*(G)) \cong H_1(G)$$

Mapping homology to K-theory

A clopen subset $A \subseteq G^0$ gives rise to a projection $1_A \in C(G^0) \subseteq C_r^*(G)$. This gives rise to a canonical homomorphism

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- Partial isometries like this live in a relative K_0 group $K_0(C(G^0) \subseteq C_r^*(G))$, so get a canonical map $H_1(G) \to K_0(C(G^0) \subseteq C_r^*(G))$

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- Factors through the desired map

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The examples are groupoids with topological property (T), which also provide counter-examples to the Baum-Connes conjecture!